STATICS

Made Simple

Fifth Edition

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Preface

In the name of Allah, the Beneficent, the Merciful

This handy book serves as an introduction to the course of Statics and is intended for first year students taking a degree or diploma in engineering. Its main objective is to provide simple and friendly techniques necessary in the learning of Statics. Focus is placed on the application of basic algebra, trigonometry and elementary calculus to solve problems with extra emphasis on the Free Body Diagram.

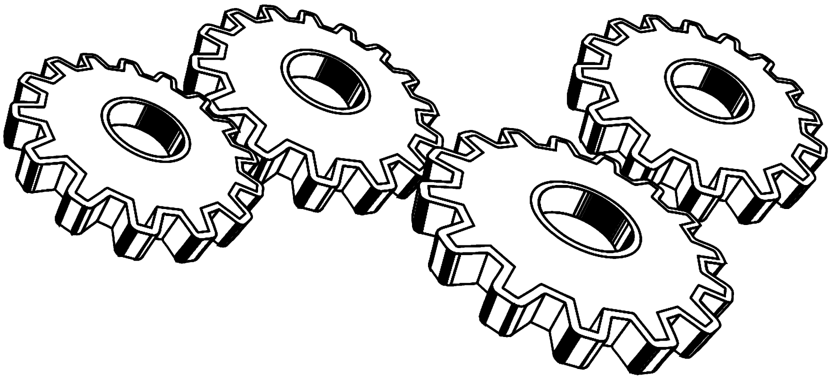
The following are some distinctive features of this book:

- Rigorous and detailed approach to solve resultant and equilibrium of particles.
- Emphasis on the techniques of drawing Free Body Diagrams.
- Thoroughly cover the moment equation to solve problems comprising statics of rigid bodies.
- Addressing various effective techniques to tackle analysis of structure problems.
- Friction topics, centroids and centre of gravities of two and three dimensional composite bodies are also included.

It is hoped that this effort, which is an attempt to guide students through a learning experience in an effective manner, will be appreciated by both lecturers and students. Any comments and suggestions for improvement are welcome and InsyaAllah will be incorporated in the next edition. The countless prior comments and suggestions made by our colleagues and students are acknowledged and highly appreciated.

We wish to express our sincere gratitude to our families, close friends, colleagues, members and ex-members of the Mechanics of Machine Panel at School of Mechanical Engineering, Universiti Teknologi Malaysia for their invaluable comments, supports and encouragements. Last but not least, we owe to Allah's grace for everything.

The Authors.
Untuk Tuhan dan Manusia



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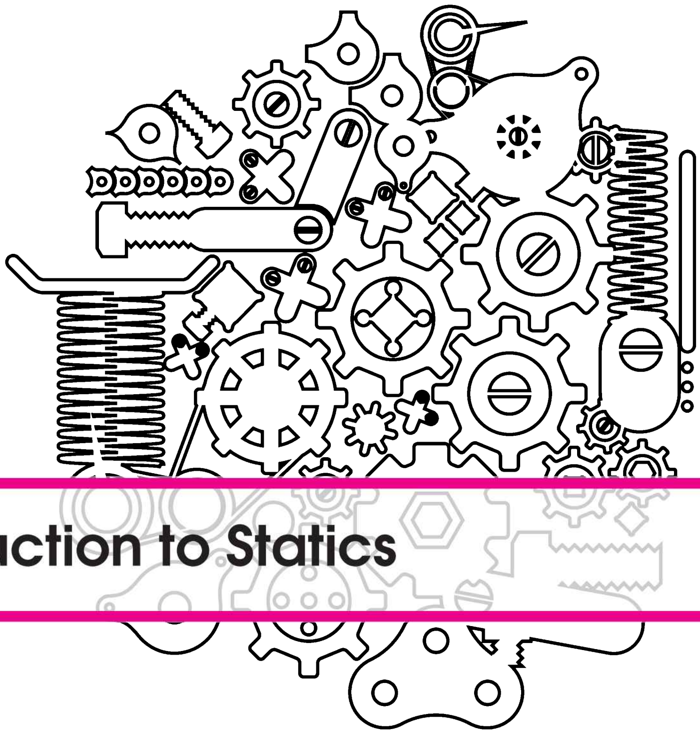


1 — Introduction to Statics

Statics comes from a broader subject which is Mechanics. Mechanics is the applied science which describes and predicts physical phenomena of bodies at rest or motion under the action of forces. Statics is focused only on the study of resultant and equilibrium. It is utilized to determine the exerted forces (external and internal) and to instigate whether a body is in equilibrium or moving with a constant velocity. In this course, we will be dealing mostly with the Newton's Three Fundamental Laws;

Concept 1.1 — Newton's Three Fundamental Laws.

First Newton Law If the resultant force acting on a particle is zero, the particle will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion).

Second Newton Law If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant force and in the same direction as the resultant force ($\sum \mathbf{F} = m\mathbf{a}$).

Third Newton Law The forces of action and reaction between bodies in contact will have the same magnitude and the same line of action but in opposite sense.

In solving Statics problems, real world objects can be modelled (simplified) as

- **Particle**
  Objects belong to particle only when its dimensions are not important (or assumed negligible) for the analysis. Thus we typically draw this type of object as a simple box, ball or point without any dimension attached to it. Imagine yourself in an air traffic control tower monitoring aircraft flying in the sky. What you see in the radar screen are just dots (or particles) representing the aircraft. Although an aircraft is big compared to yourself, it is microscopic compared to the blue sky. Thus, the omission of the aircraft dimension does not affect the calculation

results much. Hence, approximating the aircraft as particles in this case is valid whereby all the forces acting on an aircraft will be located only at a single point.

- **Rigid body**
  When dimensions of an object need to be considered in the analysis, we assumed the object to be a rigid body to simplify the calculation. A rigid body will not deform under loads, thus its dimensions remain constant and will not change with time. Even if this is not fully true in reality, such approximation can be made if we say the load is small enough such that the object will only move and not deform under the applied loading.

The customary units used in this book is the S.I. units. When solving the problem in this book, accuracy greater than 0.2% is seldom necessary or meaningful in the solution of practical engineering problems. A practical rule is to use three significant figures, except when the beginning number is one, when four significant figures are more appropriate. This rule is significant when comparing answers for two types of solutions.

Throughout the text, there will be two types of quantities which are vector quantity and scalar (magnitude only) quantity. Vector variable will be written using bold font i.e $\mathbf{F}$ while scalar variable i.e the magnitude of $\mathbf{F}$ or $|\mathbf{F}|$ is written using italic font F.

2 — Particles : Resultant 2D

2.1 Particle and rigid body

We define particles and rigid body as in Table 2.1 to simplify the analysis.

Particle	►	Possesses a mass but neglect the size (dimension)
Rigid Body	►	A combination of particles where the distance between them are the same before and after a force is acted on it and possesses a mass.

Table 2.1: Particle versus rigid body.

Generally, bodies are defined as particles when the size and shape will not significantly affect the solutions, whereby all forces are assumed to be applied at the same point.

2.2 Force vector

There are two type of physical quantities; scalar and vector quantities. The differences between them are given in Table 2.2.

Types of Quantity	**Characteristics**	**Examples**
Scalar	magnitude only	time, mass, volume
Vector	magnitude and direction	weight, force, moment

Table 2.2: Scalar and vector quantities.

Force is one of the vector quantities. A force vector has three main characteristics;

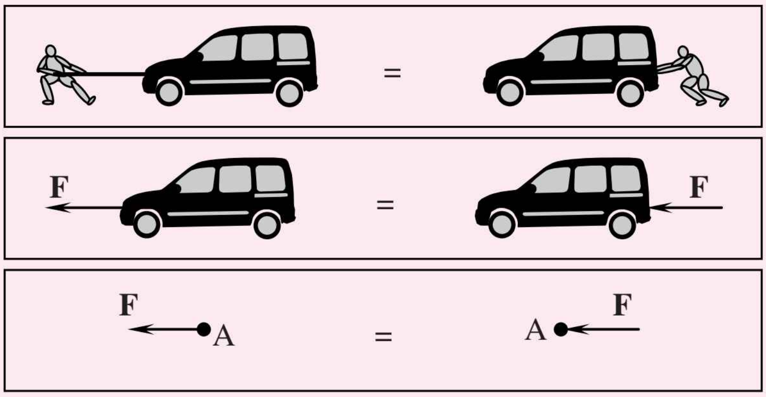
- Point of application
- Magnitude
- Direction (line of action)

Forces may change the motion of a body in two ways;

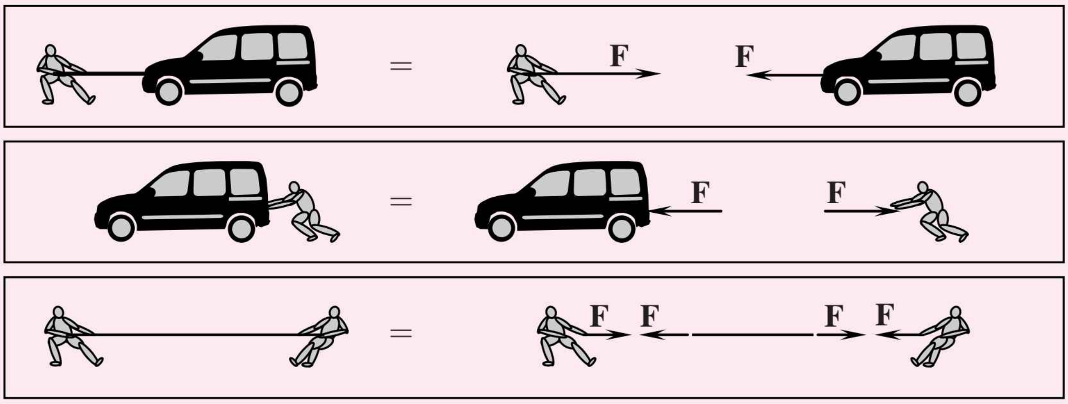
- **body at rest** → may set the body in motion.
- **body in motion** → may accelerate or retard the body motion.

When analyzing the effect of forces, concepts 2.1 and 2.2 are deduced by applying the third Newton's law given in Concept 1.1. For cases where there are multiple forces acting simultaneously on a particle or an object, Concept 2.3 aided by vector Concepts 2.4, 2.5 and 2.6 can be applied.

Concept 2.1 — Principle of transmissibility.
Pulling or pushing action produces equal force on a particle or rigid body.

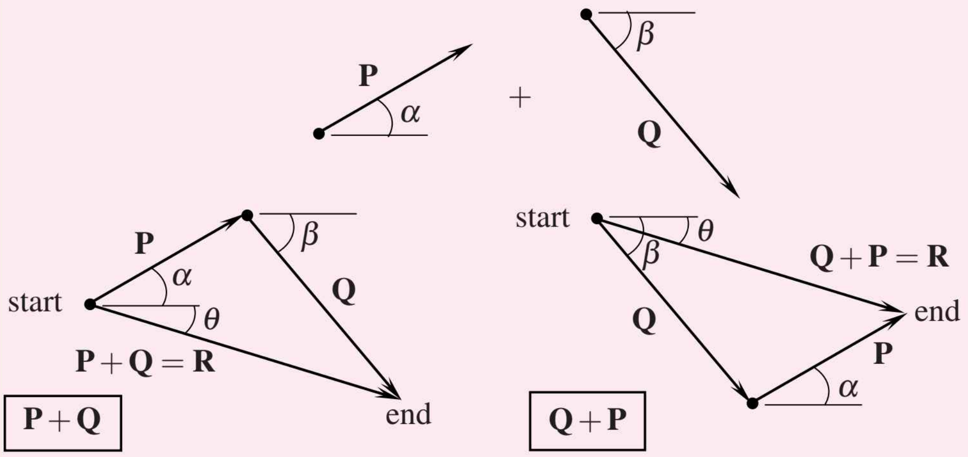


Concept 2.2 — Reaction forces.
For every acting force there is an equal and opposite reacting force.



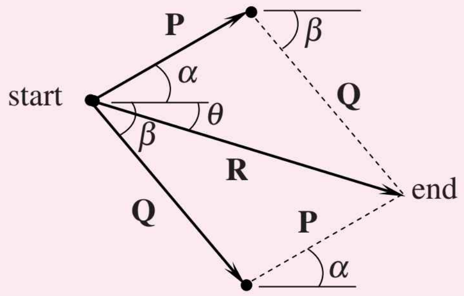
Concept 2.3 — Principle of physical independence.
If a number of forces are simultaneously acting on a particle, then each force will produce the same effect which it would have done while acting alone. Based on this principle, we can used vector operations like addition and subtraction on a force vector like a normal vector.

Concept 2.4 — Addition of two force vectors.
Vector addition needs to consider both magnitude and direction.

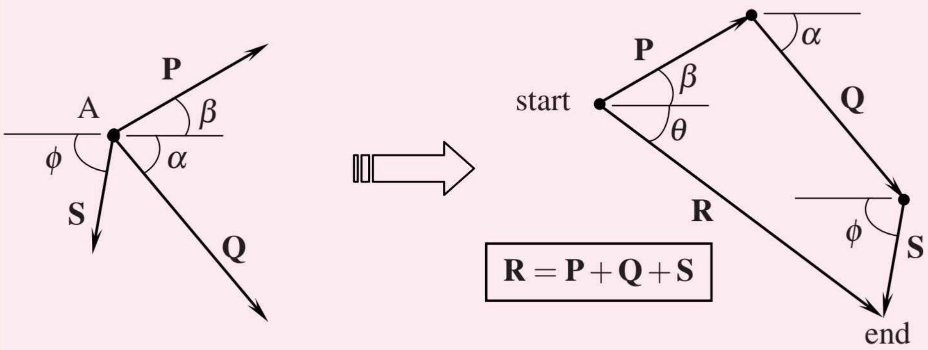


$$\mathbf{P}+\mathbf{Q}=\mathbf{Q}+\mathbf{P}=\mathbf{R}, \quad \text{where } \mathbf{R} = \text{resultant vector}$$

Concept 2.5 — Parallelogram law.
Vector operation $\mathbf{P}+\mathbf{Q}=\mathbf{Q}+\mathbf{P}=\mathbf{R}$. The order of vector addition has no effect.



Concept 2.6 — Resultant of several concurrent forces.
Resultant vector $\mathbf{R}$ is always pointing from the starting to the end point.



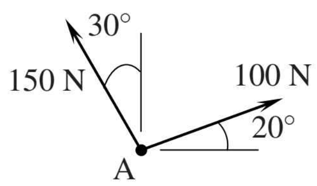
Therefore, if a number of forces **P**, **Q**, **S**, . . . are acting simultaneously on a particle, then a single force **R**, which will produce the same effect as all of the given forces, is known as a resultant force. The forces **P**, **Q**, **S**, . . . are called component forces.

2.3 Analysis approach

There are three methods to solve resultant 2D problem for particles which are;
(a) Graphical method
(b) Trigonometry method
(c) Rectangular components method

The application of these methods are best illustrated in the following example.

■ **Example 2.1** Determine the resultant of the two forces.

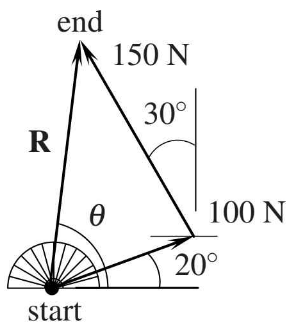


(a) Graphical method

The procedure is to start at a point and arrange the force vectors tip-to-tail. Resultant is the vector beginning from 'start' and finishing at 'end'. Observations for 2D analysis are;

- there are six quantities involved in the process of adding the two forces, i.e. the magnitude and direction of vectors 100 N, 150 N and **R**.
- except for special cases, there can be only two unknowns involved.

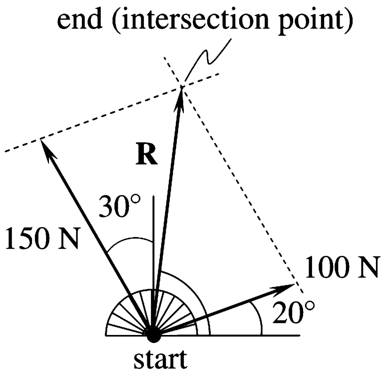
Using a scale of 1 cm $\equiv$ 50 N :
length of vector **R** $= 3.3$ cm $\equiv 165$ N $\longrightarrow$ using a ruler
$\theta = 83.5°$ $\longrightarrow$ using a protractor



In this example, the six quantities are:
100 N $\angle 20°$, 150 N $\angle 120°$ and the magnitude & direction of vector **R**.

The parallelogram law can also be applied in the graphical method to obtain similar results. The procedure is to construct two parallel lines according to the line of action of the two forces. Resultant is the vector beginning from 'start' to 'end', which is the point of intersection.

Using a scale of 1 cm $\equiv$ 50 N :
length of vector $\mathbf{R} = 3.3$ cm $\equiv 165$ N $\longrightarrow$ using a ruler
$\theta = 83.5^\circ \longrightarrow$ using a protractor



(b) Trigonometry method

The procedure is identical to the graphical method, but trigonometry is used to calculate the magnitude and direction (angle). There is no need to draw the polygon to scale. Sine and cosine rules are used. Since these rules are valid only for triangles, the number of forces involved must be three since there are represented by each side of the triangle.

$$\beta = 20^\circ + 60^\circ = 80^\circ$$

cosine rule:

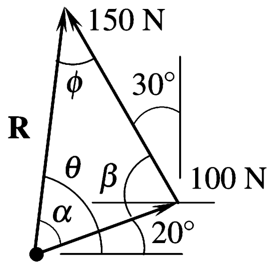
$$|\mathbf{R}|^2 = 100^2 + 150^2 - 2(100)(150)\cos 80^\circ$$
$$|\mathbf{R}| = 165.2 \text{ N}$$

sine rule:

$$\frac{\sin\alpha}{150} = \frac{\sin 80^\circ}{165.2}, \quad \alpha = 63.4^\circ$$

$$\theta = \alpha + 20^\circ = 63.4^\circ + 20^\circ = 83.4^\circ$$

$$\therefore |\mathbf{R}| = R = 165.2 \text{ N} \quad \measuredangle 83.4^\circ$$



The direction is always an angle referring to the horizontal or vertical axis.

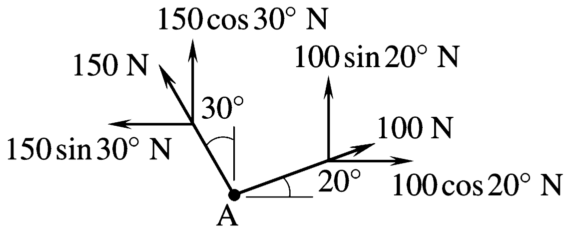
(c) Rectangular Components Method

The forces are first resolved into components along two axes which are perpendicular to each other. The norm is to use the x and y axes. By using rectangular components method, the resultant can be obtained as;

$$(\rightarrow +)\sum F_x = R_x$$

$$R_x = 100\cos 20^\circ \text{ N} - 150\sin 30^\circ \text{ N}$$

$$\therefore R_x = 18.97 \text{ N}(\rightarrow)$$



$$(\uparrow +)\sum F_y = R_y$$

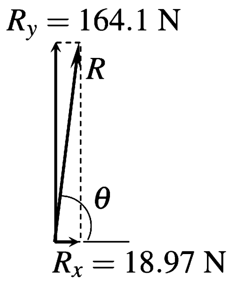
$$R_y = 100\sin 20^\circ \text{ N} + 150\cos 30^\circ \text{ N}$$

$$\therefore R_y = 164.1 \text{ N}(\uparrow)$$

$$|\mathbf{R}| = R = \sqrt{R_x^2 + R_y^2} = \sqrt{18.97^2 + 164.2^2} = 165.3 \text{ N}$$

$$\theta = \tan^{-1}\frac{164.2}{18.97} = 83.4^\circ$$

$$\therefore R = 165.3 \text{ N} \quad \measuredangle 83.4^\circ$$



Concept 2.7 — Trigonometry identities. These equations can be applied in problems involving three vectors ()

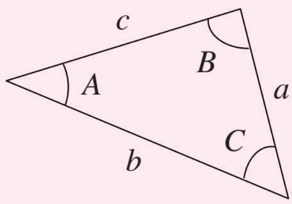
Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

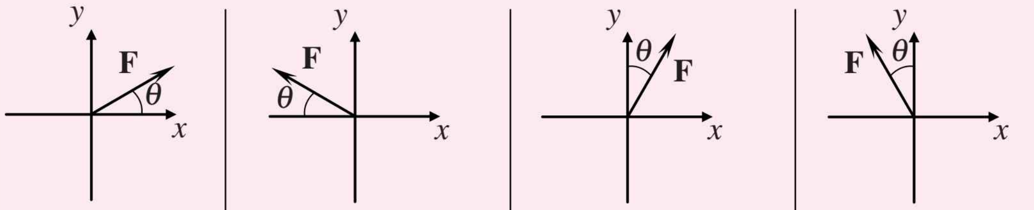
Cosine rule

$$a^2 = b^2 + c^2 - 2(bc)\cos(A)$$
$$b^2 = a^2 + c^2 - 2(ac)\cos(B)$$
$$c^2 = a^2 + b^2 - 2(ab)\cos(C)$$

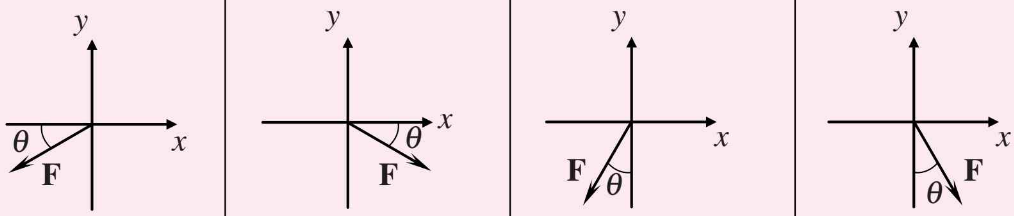
Note ambiguity case:
$\sin\theta = \sin(180^\circ - \theta)$



Concept 2.8 — Basics for resolution of forces.



$F_x = +\lvert\mathbf{F}\rvert\cos\theta$ $F_y = +\lvert\mathbf{F}\rvert\sin\theta$	$F_x = -\lvert\mathbf{F}\rvert\cos\theta$ $F_y = +\lvert\mathbf{F}\rvert\sin\theta$	$F_x = +\lvert\mathbf{F}\rvert\sin\theta$ $F_y = +\lvert\mathbf{F}\rvert\cos\theta$	$F_x = -\lvert\mathbf{F}\rvert\sin\theta$ $F_y = +\lvert\mathbf{F}\rvert\cos\theta$



$F_x = -\lvert\mathbf{F}\rvert\cos\theta$ $F_y = -\lvert\mathbf{F}\rvert\sin\theta$	$F_x = +\lvert\mathbf{F}\rvert\cos\theta$ $F_y = -\lvert\mathbf{F}\rvert\sin\theta$	$F_x = -\lvert\mathbf{F}\rvert\sin\theta$ $F_y = -\lvert\mathbf{F}\rvert\cos\theta$	$F_x = +\lvert\mathbf{F}\rvert\sin\theta$ $F_y = -\lvert\mathbf{F}\rvert\cos\theta$

Observations:

- resolutions are based on acute angles.
- the component beside the given angle is always ‘cos’.
- resolve the force first, then compare with the axes for (+ve) or (–ve) sign.

The resultant **R** is found by using;

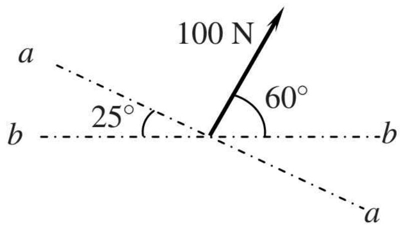
$$\mathbf{R} = \sum \mathbf{F}, \quad R_x = \sum F_x, \quad R_y = \sum F_y$$

2.4 Comparison between the methods

Main differences between the methods are:

- The graphical method is simple but might lack the accuracy required. A suitable scale must be employed to avoid errors.
- The trigonometry method is more suitable for addition of two forces. The resultant of many forces will result in longer and tedious calculations. There are two instances where this method is invaluable;
  - (a) To determine the components of a force which are not perpendicular to each other (Example 2.2).
  - (b) In special cases, when there are 3 unknowns involved, usually finding the minimum value of a force (Example 2.3).
- The rectangular components method is suitable for almost all category of questions (except the two discussed above), especially so in problems involving several concurrent forces. (Examples 2.4 and 2.5).

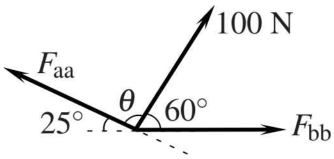
■ Example 2.2



Determine components of the 100 N force along the a − a and b − b axes.

Solution:

The 100 N force is the resultant, and must be located in between the two components (parallelogram law). i.e.

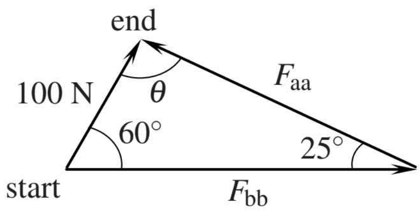


Drawing the vector triangle;

$$\theta = 180^\circ - 60^\circ - 25^\circ = 95^\circ$$

and using the sine rule

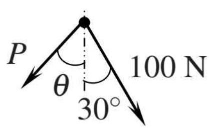
$$\frac{\sin 25^\circ}{100} = \frac{\sin 60^\circ}{F_{aa}} = \frac{\sin 95^\circ}{F_{bb}}$$



to obtain $F_{aa} = 204.9$ N and $F_{bb} = 235.7$ N

■ Example 2.3

Determine the resultant R and the minimum force P, if the resultant of the forces is vertically downwards.



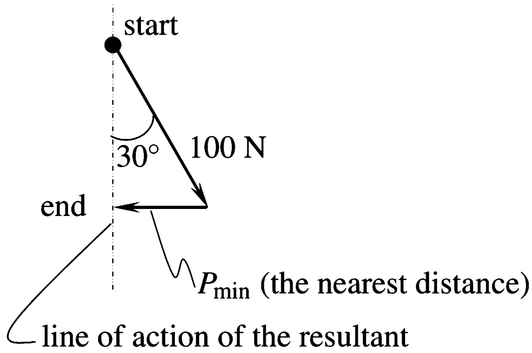
Solution:

Notice that there are 3 unknowns P, θ and R.

from the diagram,

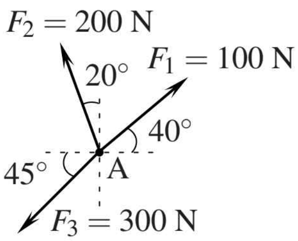
$$P = 100 \sin 30^\circ$$
$$\therefore P_{min} = 50 \text{ N}(\leftarrow)$$

$$R = 100 \cos 30^\circ$$
$$\therefore R = 86.6 \text{ N}(\downarrow)$$



■ Example 2.4

Three forces, F_1, F_2 and F_3 act on point A, determine the resultant, R.

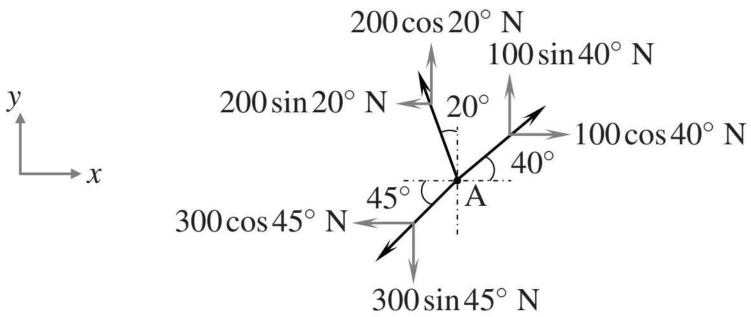


Solution:

- Observe that the two unknowns are; magnitude and direction of R.
- An assumption has to be made in the direction, usually the resultant is first resolved into components, R_x ($\rightarrow$) and R_y ($\uparrow$).

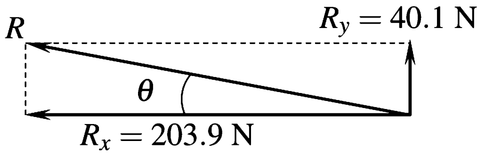
$$\mathbf{R} = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

	R	$\mathbf{F}_1$	$\mathbf{F}_2$	$\mathbf{F}_3$
magnitude	×	✓	✓	✓
direction	×	✓	✓	✓



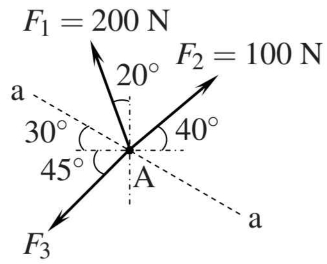
$$(+\rightarrow) \sum F_x = R_x$$
$$R_x = 100 \cos 40^\circ - 200 \sin 20^\circ - 300 \cos 45^\circ$$
$$R_x = -203.9 \text{ N}$$
$$\therefore R_x = 203.9 \text{ N}(\leftarrow)$$
$$(+\uparrow) \sum F_y = R_y$$
$$R_y = 100 \sin 40^\circ + 200 \cos 20^\circ - 300 \sin 45^\circ$$
$$R_y = 40.1 \text{ N}(\uparrow)$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{203.9^2 + 40.1^2}$$
$$= 207.8 \text{ N}$$
$$\theta = \tan^{-1}\left(\frac{40.1}{203.9}\right) = 11.13^\circ$$
$$\therefore R = 207.8 \text{ N} \quad 11.13^\circ ⦦$$



▪ Example 2.5

Three forces, F_1, F_2 and F_3 act on point A. If the resultant is known to be on the a − a axis, determine the magnitude of F_3 and the resultant R.

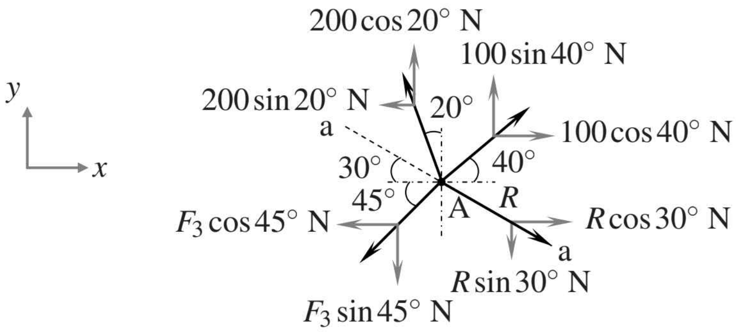


Solution 1:

- Observe that the two unknowns are the magnitudes of F_3 and R.
- The line of action of the resultant R is given, but an assumption has to be made in the direction; either, 30°⦦ or 30°⦧. For the purpose of this example 30°⦧is chosen.

$$\mathbf{R} = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

	R	**F**$_1$	**F**$_2$	**F**$_3$
magnitude	×	✓	✓	×
direction	✓	✓	✓	✓



It is not necessary to draw the above diagram in answering exam questions, but the assumption in direction of R must be stated. By solving along x and y axes;

$$
\begin{aligned}
(+\rightarrow)\sum F_x &= R_x \\
R\cos 30^\circ &= 100\cos 40^\circ - 200\sin 20^\circ - F_3\cos 45^\circ \\
0.866R &= 76.6 - 68.4 - 0.707F_3 \\
0.866R &= 8.2 - 0.707F_3 \quad \ldots(1) \\
(+\uparrow)\sum F_y &= R_y \\
R\sin 30^\circ &= 100\sin 40^\circ + 200\cos 20^\circ - F_3\sin 45^\circ \\
-0.5R &= 64.3 + 187.9 - 0.707F_3 \\
-0.5R &= 252.2 - 0.707F_3 \quad \ldots(2)
\end{aligned}
$$

solving for R and F_3

$$
\begin{aligned}
(1)-(2) \quad 1.366R &= -244\ \text{N} \\
R &= -178.6\ \text{N} \\
\therefore R &= 178.6\ \text{N} \quad 30^\circ \angle \text{ which is opposite to the assumed direction}
\end{aligned}
$$

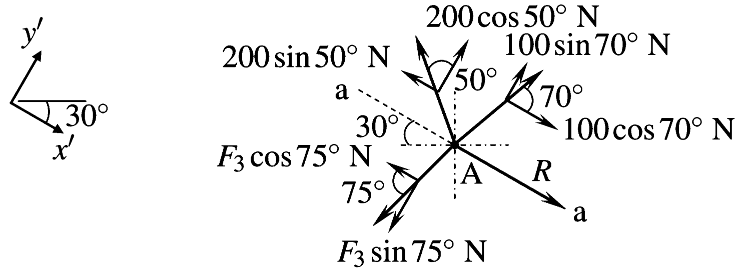
input the value of R into (1)

$$
\begin{aligned}
0.866(-178.6) &= 8.2 - 0.707F_3 \\
F_3 &= 230.4\ \text{N} \quad 45^\circ \angle
\end{aligned}
$$

Observe that the value of R used is the one resulting from calculations (-178.6 N) and not the modified answer (178.6 N).

Solution 2:

Another method is to use slanting axes where the forces are resolved through 90° components that are not vertical and horizontal. Usually, the axes is positioned so that one axis is on an unknown quantity (i.e. R)



The other unknown (F_3) is directly solved by summation of forces in the y' direction;

$$(+\nearrow)\sum F_{y'} = R_{y'} \quad R_{y'} = 0 \text{ since } R \text{ is in the } x' \text{ direction.}$$
$$0 = 100\sin 70^\circ + 200\cos 50^\circ - F_3 \sin 75^\circ$$
$$\therefore F_3 = 230.4 \text{ N} \quad 45^\circ$$

R is solved by the summation of forces in the x' direction;

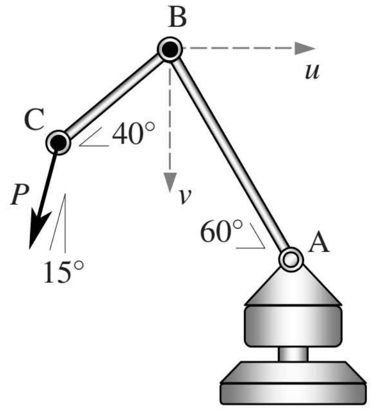
$$(+\searrow)\sum F_{x'} = R_{x'} \quad R_{x'} = R \text{ since } R \text{ is in the } x' \text{ direction.}$$
$$R = 100\cos 70^\circ - 200\sin 50^\circ - F_3\cos 75^\circ$$
$$R = 100\cos 70^\circ - 200\sin 50^\circ - (230.4)\cos 75^\circ$$
$$R = -178.6 \text{ N}$$
$$\therefore R = 178.6 \text{ N} \quad 30^\circ$$

▪ Example 2.6

Force P acts on the robot arm as shown.

(a) If the component of force P along BC equals 500 N, determine the force P and component of force P parallel to the arm AB.

(b) Determine the rectangular components of the 500 N force (component of force P along BC) about the u and v axes.



Solution:

(a) The force **P** has component along AB and BC. Hence;

	$\mathbf{P}_{BC}$	+	$\mathbf{P}_{AB}$	=	$\mathbf{P}$
magnitude	500		P_{AB}		P
direction	40°		60°		75°

$$(+\rightarrow)\sum F_x = R_x$$
$$-500\cos 40^\circ + P_{AB}\cos 60^\circ = -P\cos 75^\circ$$
$$-383 + 0.5P_{AB} = -0.259P \quad \ldots(1)$$
$$(+\uparrow)\sum F_y = R_y$$
$$-500\sin 40^\circ - P_{AB}\sin 60^\circ = P\sin 75^\circ$$
$$-321.4 - 0.866P_{AB} = -0.966P \quad \ldots(2)$$

Solving equations (1) and (2) simultaneously gives $P = 696$ N and $P_{AB} = 405.6$ N.

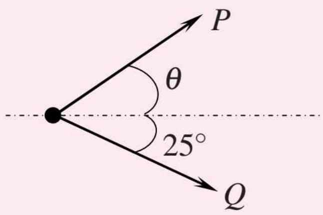
(b) The u and v axes are along horizontal and vertical directions respectively. Hence the component of force $\mathbf{P}_{BC}$ along these axes are;

$$(+\rightarrow) \quad P_{BCu} = -500\cos 40^\circ = -383 \text{ N}$$
$$(+\downarrow) \quad P_{BCv} = 500\sin 40^\circ = 321 \text{ N}$$

2.5 Example questions

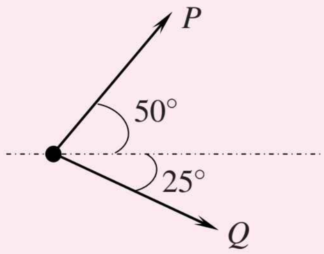
Exercise 2.1

Force $Q = 80$ N and the resultant $R = 120$ N ($\rightarrow$). Determine the magnitude of force P and the angle θ.
Answers: $P = 58.3$ N, $\theta = 35.44°$



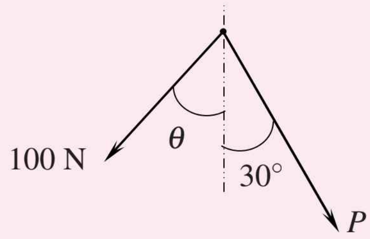
Exercise 2.2

Force $Q = 80$ N and the resultant R is horizontal. Determine the magnitudes of both forces P and R.
Answers: $P = 44.14$ N, $R = 100.9$ N



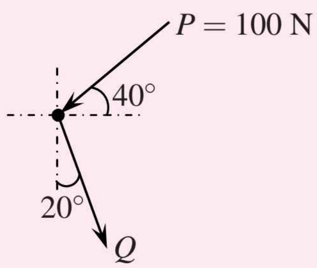
Exercise 2.3

a. Determine P and θ if the resultant $R = 160$ N ($\downarrow$).
b. If R is vertical and $\theta = 40°$, find forces P and R.

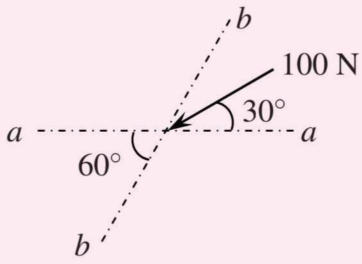


Exercise 2.4

a. The resultant R of forces P and Q shown acts along the vertical axis. Determine the magnitude of forces Q and R.
b. The resultant R of forces P and Q shown is of magnitude 200 N. Determine the magnitude of force Q and direction of force R.

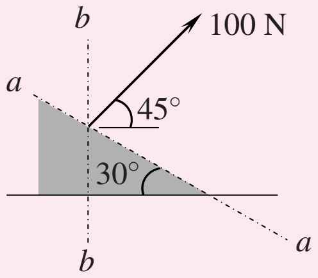


Exercise 2.5



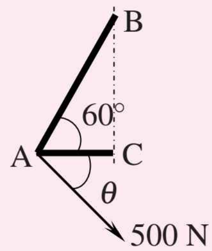
Determine components of the 100 N force shown along the *a*-*a* and *b*-*b* axes.
Answers: 86.6 N for both axes

Exercise 2.6



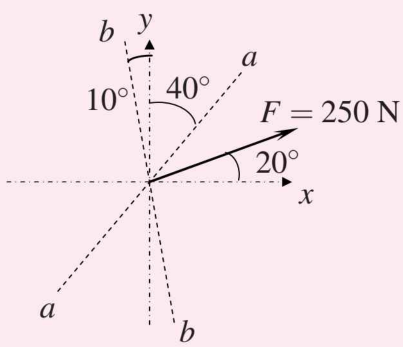
Determine components of the $F = 100$ N force shown along the *a*-*a* and *b*-*b* axes.

Exercise 2.7



The 500 N force is applied to the bent frame BAC. If the component along AC is 400 N, determine the angle θ and the force component along AB.
Answers: $F_{AB} = 560.6$ N and $\theta = 76.15°$

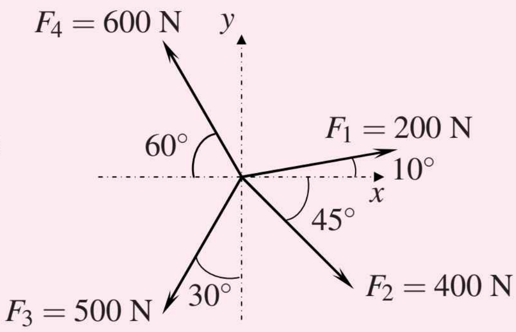
Exercise 2.8



Determine the components of the 250 N force shown along the axes *a*-*a* and *b*-*b*.

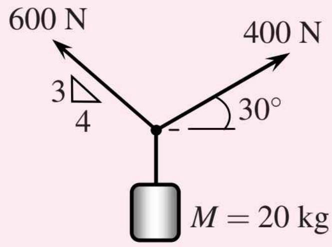
Exercise 2.9

Determine the resultant of the four forces shown.



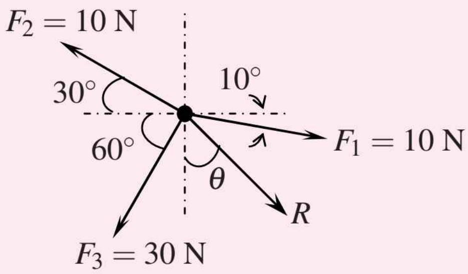
Exercise 2.10

a. Determine the resultant force for the system shown.
b. Determine the change in the mass M if the resultant force is horizontal.



Exercise 2.11

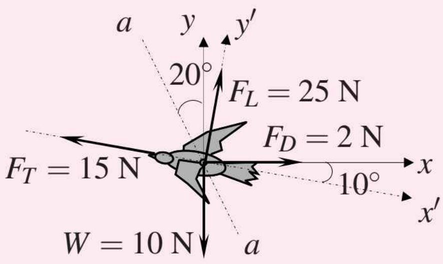
Resultant of the three forces is assumed in the direction shown. Determine the magnitude of resultant R and the angle θ.



Exercise 2.12

Four forces act on a bird in flight, as shown in the figure; its weight W, the thrust F_T, the lift F_L provided by the wings, and the drag F_D resulting from its motion through air. Determine

a. the resultant of the four forces and its line of action with respect to the x axis.
b. the components of the resultant in the a-a and x'-x' directions.



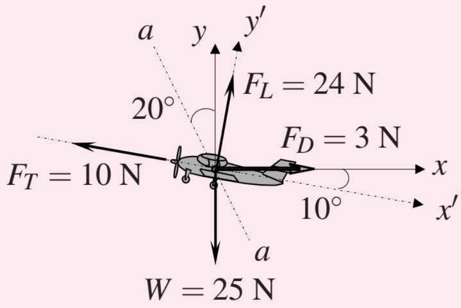
Answers: (a) $R = 19.17$ N, $\theta = 63.91°$, (b) $R_{(a-a)} = 17.9$ N, $R_{(x'-x')} = 2.35$ N

Exercise 2.13

Four forces act on a plane in flight, as shown in the figure; its weight W, the thrust F_T, the lift F_L provided by the wings, and the drag F_D resulting from its motion through air. Determine

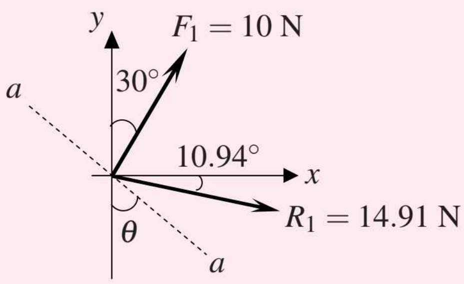
a. the resultant of the four forces and its line of action with respect to the x' axis.

b. the components of the resultant in the a-a and y'-y' directions.



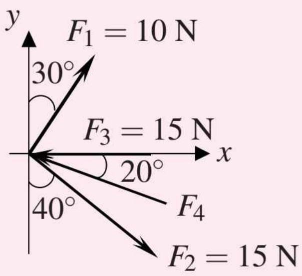
Exercise 2.14

R_1 is the resultant for F_1 and F_2. If the force F_2 acts along a-a, determine F_2 and the angle θ.
Answer: $F_2 = 15$ N and $\theta = 40°$



Exercise 2.15

If the component of the resultant R (for forces F_1, F_2, F_3 and F_4) along the x axis is 0.59 N, determine F_4 and the resultant R.

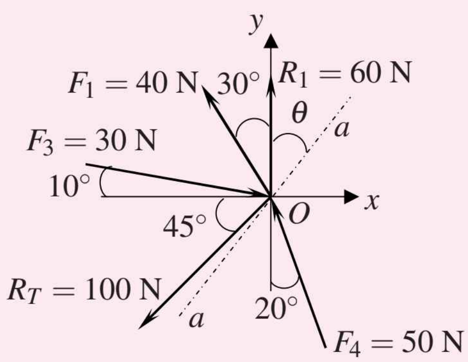


Exercise 2.16

Five forces, F_1, F_2, F_3, F_4 and F_5 are applied to point O. The force F_2 which acts along a-a, and F_5 are not shown.

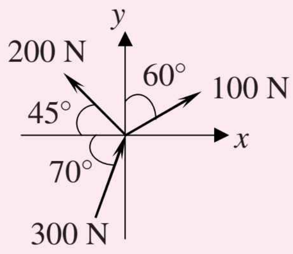
a. Find the magnitude of F_2 and the angle θ if R_1 is the resultant of F_1 and F_2.

b. Find F_5 if R_T is the resultant for F_1, F_2, F_3, F_4 and F_5.



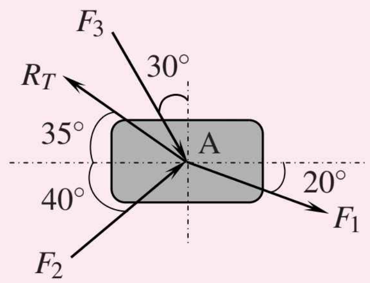
Exercise 2.17

a. Determine the resultant for the three forces shown.
b. If there are four forces applied to the point, find the fourth force if the resultant for the four forces is zero.



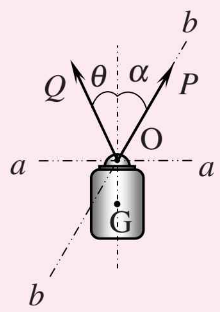
Exercise 2.18

Four forces act on point A with details as follows; $F_1 = 0.8$ kN, $F_2 = 1.9$ kN, $F_3 = 0.15$ kN and F_4 is not shown.
a. Find the resultant for F_1, F_2 and F_3.
b. If resultant for the four forces is $R_T = 2$ kN in the direction shown, determine F_4.



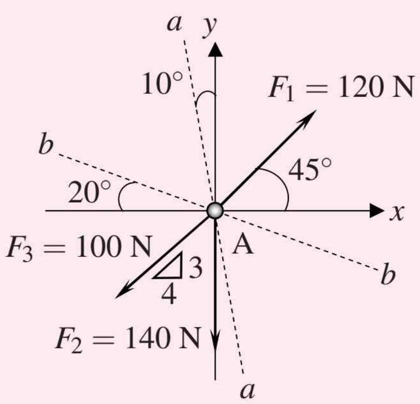
Exercise 2.19

Two forces, P and Q are applied at O to counter a 500 N load with centre of gravity at G. Determine
a. the two components of Q if it is resolved along a-a and b-b axes, given that $Q = 250$ N, $\theta = 25°$ and $\alpha = 30°$.
b. the minimum force Q and the angle θ if $P = 400$ N, $\alpha = 60°$ and the resultant R for the whole system is vertical.

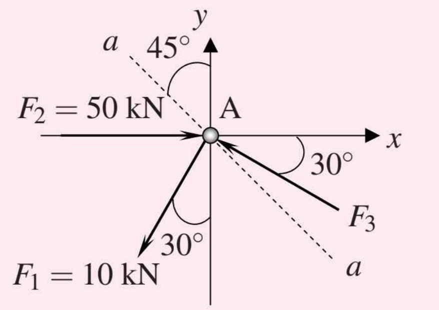


Exercise 2.20

Three forces F_1, F_2 and F_3 are applied to point A. Determine the two forces, acting on a-a and b-b axes which is equivalent to the resultant of the three forces.



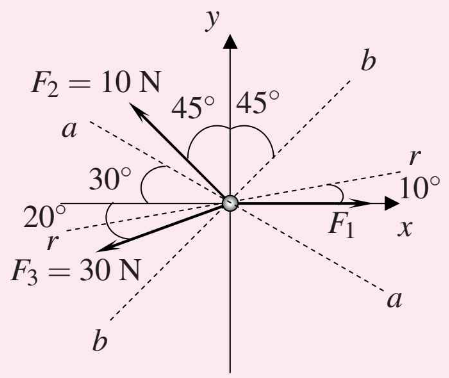
Exercise 2.21



Resultant of the three forces F_1, F_2 and F_3 acts along *a-a*. Determine

a. magnitude of the force F_3.
b. the resultant R.

Exercise 2.22

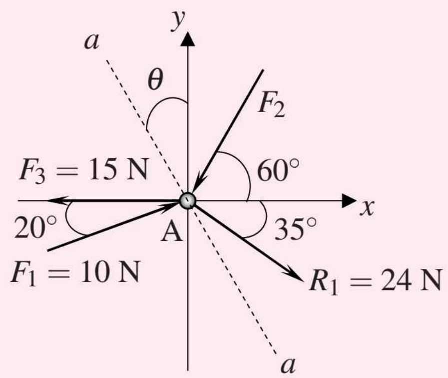


Resultant of the three forces F_1, F_2 and F_3 acts along *r-r*. Determine the magnitude of F_1 and the resultant R. Also, determine the two components if R is resolved along *a-a* and *b-b*.

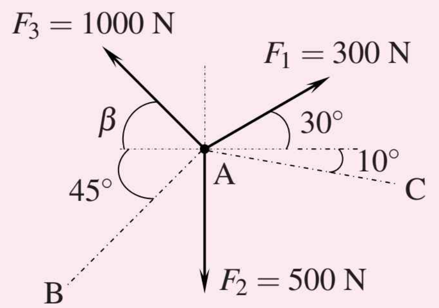
Exercise 2.23

Four forces; F_1, F_2, F_3 and F_4 are applied to point O. The force F_4 acts along *a-a*.

a. If $R_1 = 24$ N is the resultant for F_1 and F_4, determine the magnitude of F_4 and the angle θ.
b. If the component of the resultant (for F_1, F_2, F_3 and F_4) along the x axis is -8.1 N, determine magnitude of F_2 and the resultant R.



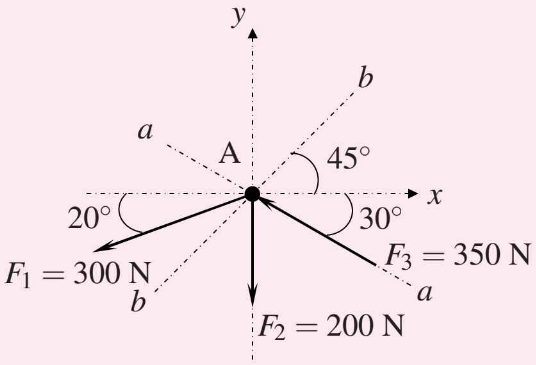
Exercise 2.24



a. If resultant of the three forces is of magnitude 750 N, determine angle β and direction of the resultant.
b. Find the two components if F_1 is resolved along AB and AC.

Exercise 2.25

The resultant of four forces F_1, F_2, F_3 and F_4 (not shown) acting on particle A lies on the line *a-a*. Determine the magnitudes of the resultant and F_4 if F_4 acts on the line *b-b*.

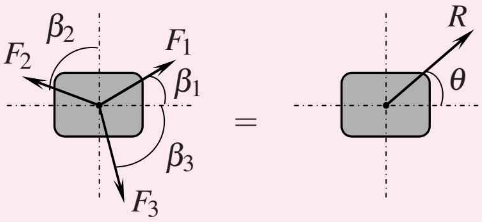


Exercise 2.26

Solve for R and θ given the following equations;

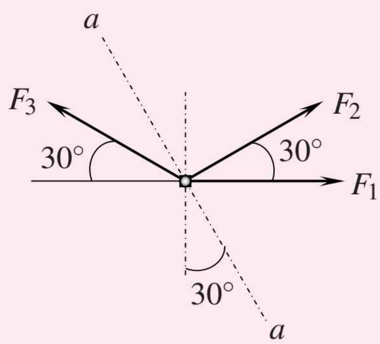
$$(+\rightarrow)\ R\cos\theta = F_1\cos\beta_1 - F_2\sin\beta_2 + F_3\cos\beta_3 = 0\ \text{N}$$

$$(+\uparrow)\ R\sin\theta = F_1\sin\beta_1 - F_2\cos\beta_2 - F_3\sin\beta_3 = 5\ \text{N}$$



Exercise 2.27

It is required that resultant of the three forces, $F_1 = 100$ N, $F_2 = 200$N and $F_3 = 300$ N acts on the line *a–a*. While maintaining F_1 and F_2 (magnitude and direction), determine magnitude of the resultant and the additional force to F_3 to satisfy the requirement. The direction for force F_3 is unchanged.



3 — Particles : Equilibrium 2D

3.1 Equilibrium condition

A particle is in equilibrium when the resultant of all forces acting on it equals zero. Hence when a particle is in equilibrium condition;

$$\sum \mathbf{F} = \mathbf{R} = 0 \quad \Rightarrow \quad \therefore \quad \text{their components } \sum F_x = R_x = 0 \quad \text{and} \quad \sum F_y = R_y = 0$$

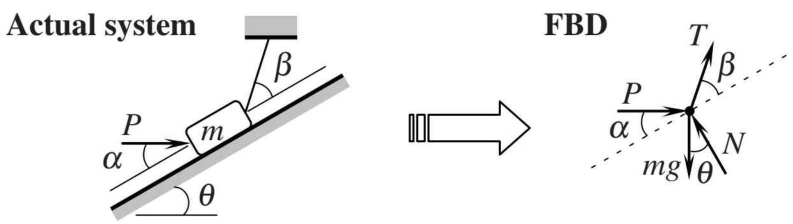
3.2 Free body diagram (FBD)

All relevant forces acting on any particle can be identified by drawing the Free–Body Diagram (FBD) of the particle.

The procedure of drawing the FBD is as follows;

- Draw the boundary of the chosen section and detach / separate it from all other particles,
- Input all external forces acting on the particle,
- For a particle with a mass, put the weight, $W = mg$ acting at the centre of gravity (G) of the particle in the vertically downwards direction,
- For a particle touching a surface there is a normal force acting perpendicular to the surface and directing towards the particle.

Example 3.1



3.3 Analysis approach

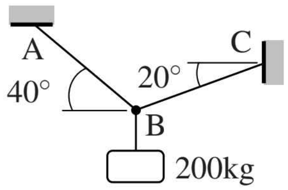
There are three methods to solve equilibrium 2D problem for particles which are;

(a) Graphical method
(b) Trigonometry method
(c) Rectangular components method

The application of these methods are best illustrated in the following example.

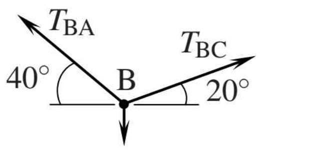
▪ Example 3.2

Determine the tension in cables AB and BC for the system to be in equilibrium.



Solution:

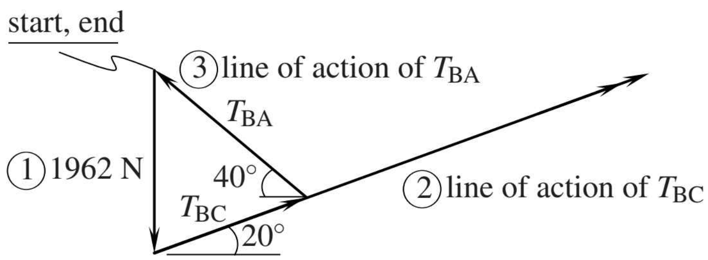
Preliminary step → Draw FBD



FBD

A sketch of the FBD is a **MUST** when answering all questions involving equilibrium.

(a) Graphical Method



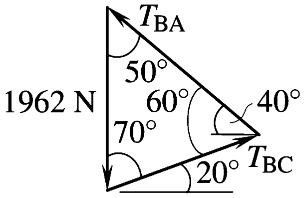
For equilibrium, the end result is a closed polygon (the 'start' and 'end' is at the same point). The procedure of this method is the following;

- Start at a point and construct a line with a suitable scale to represent a force with a known magnitude and direction, in this case the 1962 N force acting vertically downwards.
- Construct a line to represent a known line of action (say choose T_{BC} at 20°) starting from the end point of (1).
- The other line of action (T_{BA} at 40°) must pass through the above line and end at the start point, completing the closed polygon.

Using a suitable scale, the result is found to be
$T_{BA} = 2120$ N and $T_{BC} = 1735$ N

(b) Trigonometry Method

The procedure for this method is similar to the graphical method except that trigonometric identities are used to determine the magnitude of T_{BA} and T_{BC}.



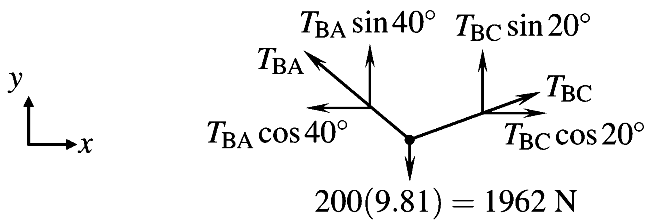
sine rule:

$$\frac{1962}{\sin 60^\circ} = \frac{T_{BA}}{\sin 70^\circ} = \frac{T_{BC}}{\sin 50^\circ}$$

$\therefore T_{BA} = 2128.89$ N and $T_{BC} = 1735.49$ N

(c - i) Rectangular Components Method ($x - y$ axis)

All forces are resolved into their x and y components

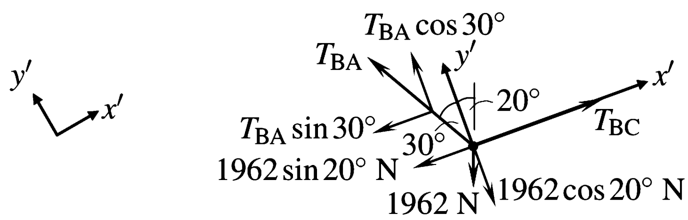


$(+\rightarrow) \sum F_x = 0$
$T_{BC} \cos 20^\circ - T_{BA} \cos 40^\circ = 0$
$\therefore T_{BC} = 0.8152 T_{BA}$

$(+\uparrow) \sum F_y = 0$
$T_{BC} \sin 20^\circ + T_{BA} \sin 40^\circ - 1962 = 0$
Substituting for T_{BC} gives;
$0.8152 T_{BA} \sin 20^\circ + T_{BA} \sin 40^\circ - 1962 = 0$
$\therefore T_{BA} = 2128.87$ N $\quad T_{BC} = 1735.46$ N

(c - ii) Rectangular Components Method (slanting axis)

Another (not so popular) method is to resolve the forces through 90° components that are not horizontal and vertical. All forces are resolved into their x' and y' components. Usually, the axis is positioned so that one axis is on an unknown quantity (T_{BC}).



Observe that the other unknown (T_{BA}) is directly solved by summation of forces in the y' direction.

$$(+\nwarrow)\sum F_{y'} = 0$$
$$= -1962\cos 20^\circ \text{ N} + T_{BA}\cos 30^\circ = 0$$
$$\therefore T_{BA} = 2129 \text{ N}$$

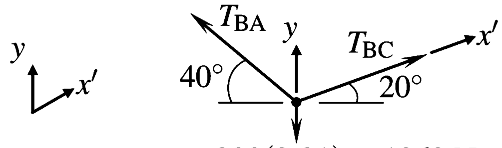
T_{BC} is solved by the summation of forces in the x' direction.

$$(+\nearrow)\sum F_{x'} = 0$$
$$= T_{BC} - 1962\sin 20^\circ \text{ N} - T_{BA}\sin 30^\circ = 0$$
$$= T_{BC} - 1962\sin 20^\circ \text{ N} - 2129\sin 30^\circ = 0$$
$$\therefore T_{BC} = 1735 \text{ N}$$

Advantage	Disadvantage
Directly solve one unknown	Must be good in determining angles

(c - iii) Rectangular Components Method (independent axes)

Another (an even less popular) method is to resolve the forces through axes that are not 90° to each other. This method is seldom used but it is shown here just to emphasize that, as long as the resolutions to components are done correctly, choice of axes is irrelevant.



$$(+\nearrow)\sum F_{x'} = 0$$
$$T_{BC} - 1962\sin 20^\circ \text{ N} - T_{BA}\sin 30^\circ = 0$$
$$T_{BC} - 1962\sin 20^\circ \text{ N} - 2129\sin 30^\circ = 0$$
$$\therefore T_{BC} = 1735 \text{ N}$$

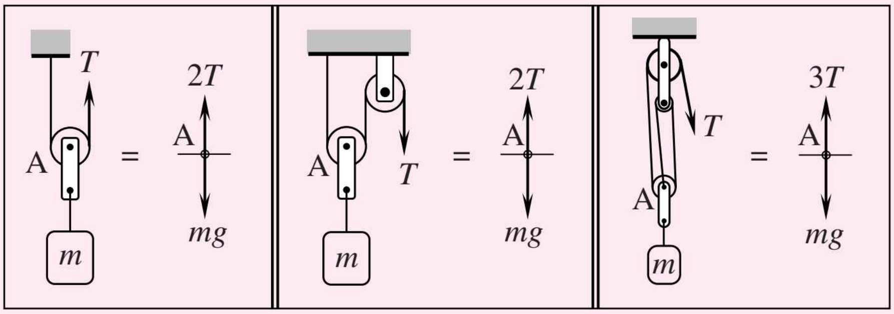
$$(+\uparrow)\sum F_{y} = 0$$
$$T_{BC}\sin 20^\circ + T_{BA}\sin 40^\circ - 1962 \text{ N} = 0$$
$$1735\sin 20^\circ + T_{BA}\sin 40^\circ - 1962 \text{ N} = 0$$
$$\therefore T_{BA} = 2129 \text{ N}$$

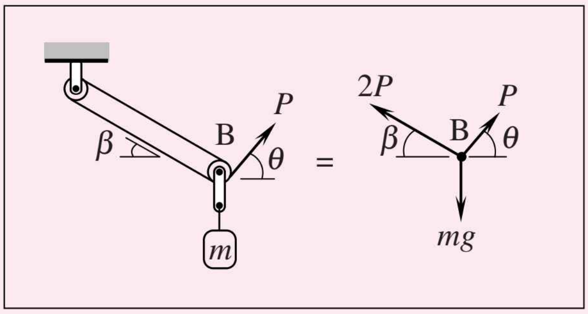
Advantage	Disadvantage
Free to choose axes	Making sure the resolutions are correct to their respective chosen axis

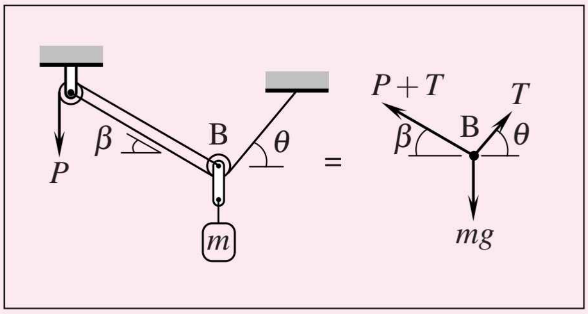
Any other chosen axes will generate the **SAME** results !

Concept 3.1 — Analysis of cables/wires/ropes and pulleys.

- All cables are assumed to be inextensible.
- All forces acting from cables must direct outwards from point of analysis, i.e. in tension.
- When a cable passes a pulley, the tension is the same as long as it is the same cable.
- Pulleys are assumed to be smooth except stated otherwise.
- Dimensions of a pulley are usually neglected in calculations except stated otherwise.







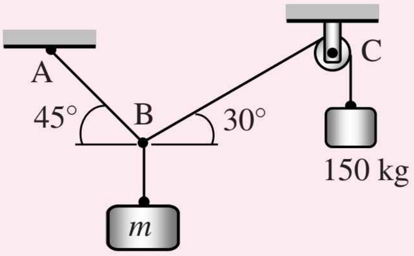
3.4 Example questions

Exercise 3.1

The system shown in the figure is in equilibrium. Determine

a. tension in cables AB and BC.
b. the mass m.

Answer: (a) $T_{AB} = 1802$ N, $T_{BC} = 1471.5$ N
(b) $m = 204.9$ kg

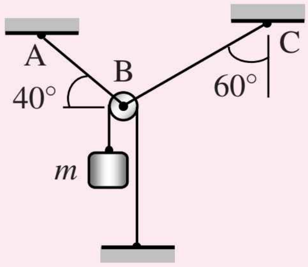


Exercise 3.2

If the tension in cable BC is 800 N, determine.

a. tension in cables AB.
b. the mass m.

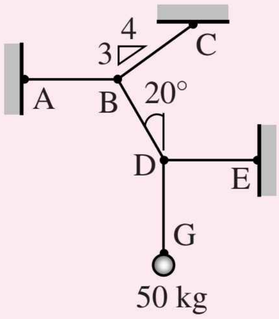
Answer: (a) $T_{AB} = 904.4$ N (b) $m = 50$ kg



Exercise 3.3

The 50 kg mass is supported by a five cable system. Determine the tension in every cable so that the system is in equilibrium.

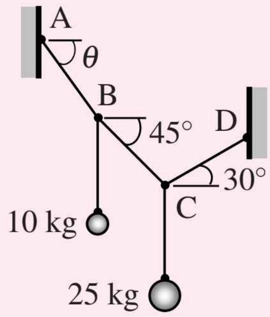
Answer: $T_{AB} = 832.5$ N, $T_{BD} = 522$ N,
$T_{DG} = 490.5$ N, $T_{BC} = 817.5$ N and $T_{DE} = 178.5$ N



Exercise 3.4

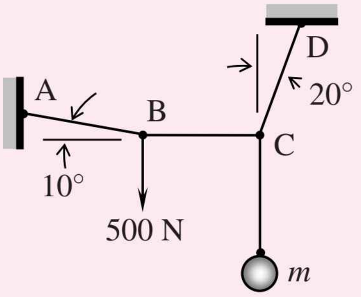
The system shown in the figure is in equilibrium. Determine

a. the tension in cable AB.
b. the angle θ.



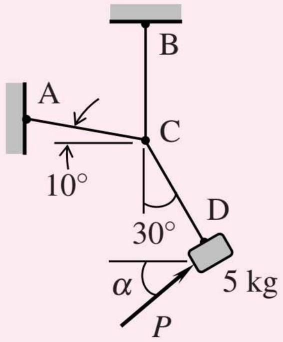
Exercise 3.5

The cable system is used to maintain the position of mass m. If a 500 N force is acting at B as shown, determine the tension in cables BC, CD and the mass m.



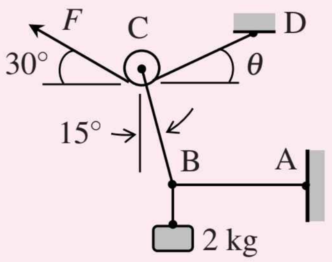
Exercise 3.6

If the tension in cable AC is 20 N, determine the tension in cable BC. Determine also the magnitude and direction of force P so that equilibrium is maintained.



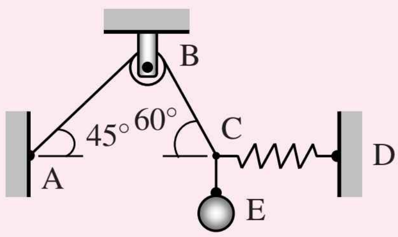
Exercise 3.7

Determine the force F and angle θ if the system shown is in equilibrium.
Answer: $F = 14.36$ N and $\theta = 60°$



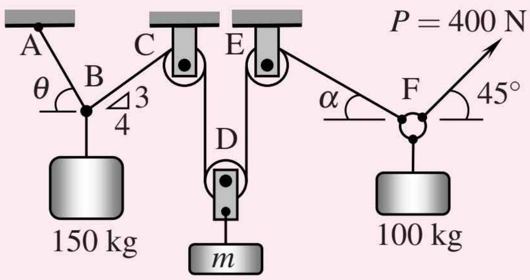
Exercise 3.8

A 6 kg mass at E is supported as shown. Determine the tension in the spring and the tension in cable AB.



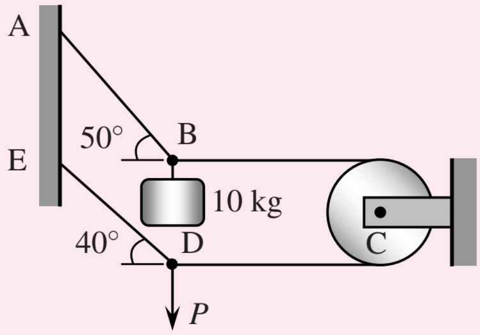
Exercise 3.9

The figure shows a system of pulleys supporting several masses. Determine the mass m and angle θ so that equilibrium is maintained.



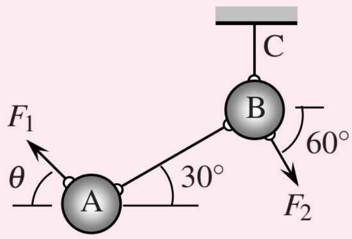
Exercise 3.10

The force P is acting at point D to maintain the system in equilibrium. Determine the magnitude of force P.



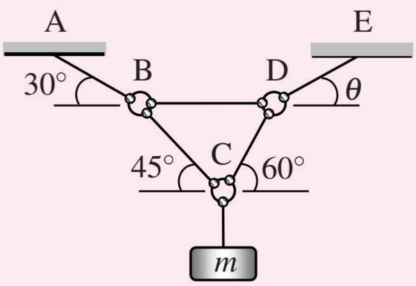
Exercise 3.11

Two forces, F_1 and F_2 are used to maintain the position of both 50 kg mass balls as shown in the figure. If the tension in cable BC is found to be 2000 N, determine F_1, F_2 and angle θ.



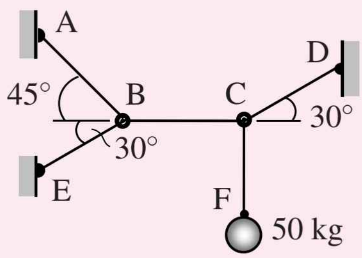
Exercise 3.12

The mass $m = 30$ kg is supported by six cables as shown. Determine the tension in horizontal cable BD and angle θ to maintain equilibrium.



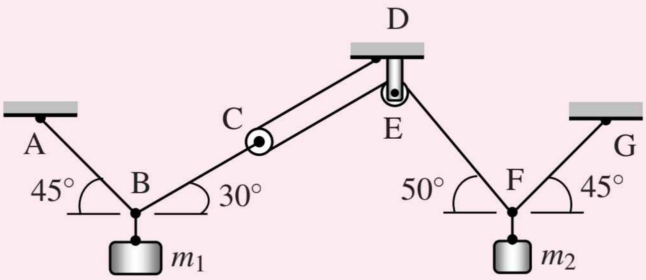
Exercise 3.13

The 50 kg mass is supported by a five cable system as shown. Determine the tension in all cables.



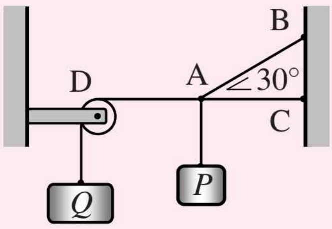
Exercise 3.14

Determine the mass m_1 used to maintain mass $m_2 = 10$ kg at the position shown.
Answer: $m_1 = 19.4$ kg



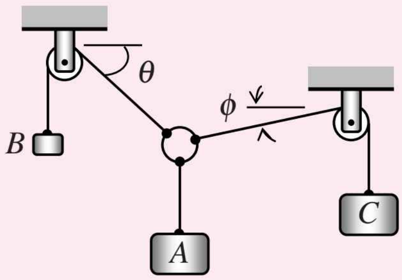
Exercise 3.15

Two masses, P and Q are used to maintain DAC in the horizontal position as shown. If $P = 500$ kg and $Q = 1000$ kg, determine the tension in cables AB, AC and AD.



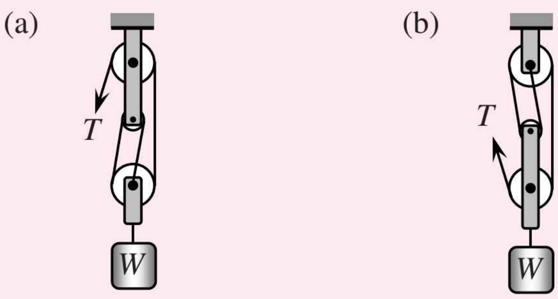
Exercise 3.16

The three loads are supported by the cable and pulley system. If the weight A equals W, weight B equals $0.25W$ and weight C equals W, determine the equilibrium angle θ.



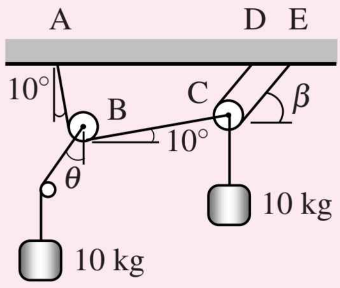
Exercise 3.17

Determine the tension in the cable T in terms of the weight W.



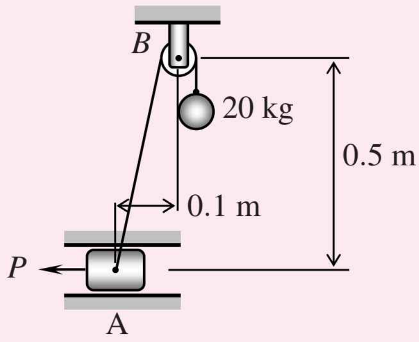
Exercise 3.18

The system is in equilibrium. Determine the angles θ and β, and tension in all cables.



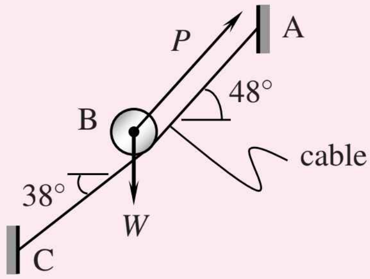
Exercise 3.19

Collar A is connected to the 20 kg mass and moves freely along the slot. Determine the force P required to maintain the collar in equilibrium.



Exercise 3.20

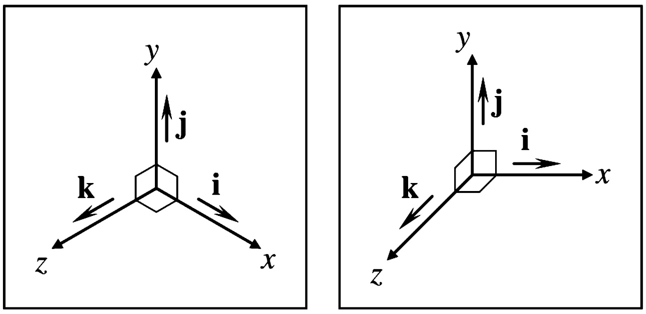
The system shown is in equilibrium. If $P = 20$ kN, determine the load W.



4 — Particles : Resultant 3D

4.1 Forces in space

In space, we are dealing with three dimensional coordinate system. Here only the Cartesian coordinate is considered. This coordinate system is made up by three perpendicular axes namely x, y and z axes. Two common arrangements for these axes are shown below;

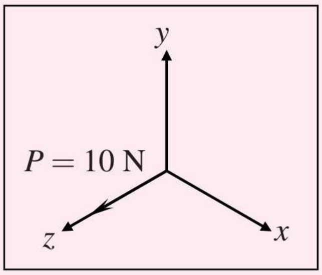


For a particular force vector in space, there are also three components of this force which correspond to each of the respective axis. In order to determine these force components, three methods are available;

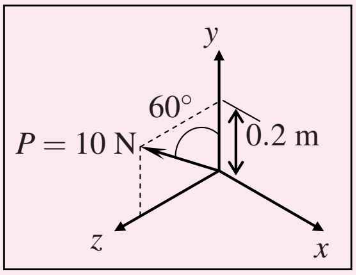
- First angle method
- Second angle method
- Coordinate method

Concept 4.1 Force vector in space (3D)

Before that...some basic, simple 'must know' is how to determine the force **P** in terms of unit vectors **i**, **j** and **k**.

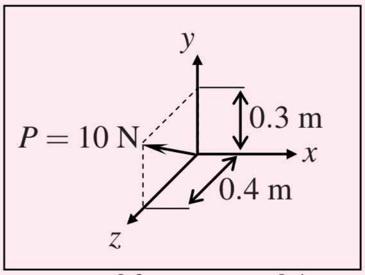


$\mathbf{P} = 10\text{N}\mathbf{k}$



$\mathbf{P} = 10(\cos 60°\text{N}\mathbf{j} + \sin 60°\text{N}\mathbf{k})$

$\mathbf{P} = 5\text{N}\mathbf{j} + 8.66\text{N}\mathbf{k}$



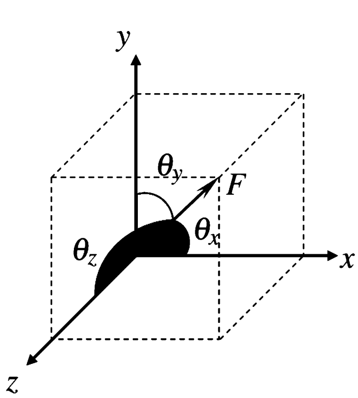
$\mathbf{P} = 10(\frac{0.3}{0.5})\text{N}\mathbf{j} + 10(\frac{0.4}{0.5})\text{N}\mathbf{k}$

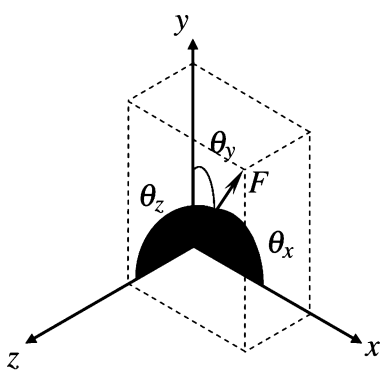
$\mathbf{P} = 6\text{N}\mathbf{j} + 8\text{N}\mathbf{k}$

Determine the resultant vector for displacement using Pythagoras Theorem first.

4.2 First angle method

This method is suitable when angles between the force and the three axes are known/given. Consider a force vector **F** with magnitude F and at angles θ_x, θ_y and θ_z to x, y and z axes, respectively.





This typical force vector can be written in the form of;

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

However, based on the figure above, its components are given by;

$$F_x = F\cos\theta_x$$
$$F_y = F\cos\theta_y$$
$$F_z = F\cos\theta_z.$$

Thus, the force vector can be rewritten as;

$$\begin{aligned}\mathbf{F} &= F\cos\theta_x\mathbf{i} + F\cos\theta_y\mathbf{j} + F\cos\theta_z\mathbf{k}\\ &= F(\cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{j} + \cos\theta_z\mathbf{k}).\end{aligned}$$

But,

$$\cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{j} + \cos\theta_z\mathbf{k} = \lambda$$

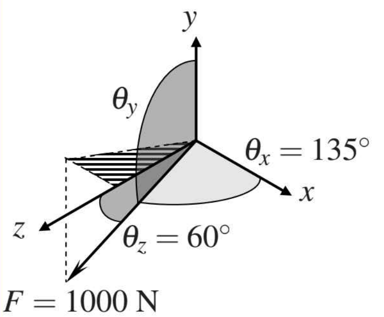
where λ is a unit vector along the line of action of the force vector $\mathbf{F}$. Therefore, force vector $\mathbf{F}$ can also be written in the form of;

$$\therefore \mathbf{F} = F\lambda = F\lambda_x\mathbf{i} + F\lambda_y\mathbf{j} + F\lambda_z\mathbf{k}$$

	Vector Identities
1.	$\lambda_x = \cos\theta_x = \frac{F_x}{F}$; $\lambda_y = \cos\theta_y = \frac{F_y}{F}$; $\lambda_z = \cos\theta_z = \frac{F_z}{F}$
2.	$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1$ $\therefore \cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$
3.	$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$

■ Example 4.1

Determine the 1000 N force in terms of **i**, **j** and **k**.



Solution:

Determine θ_y

$$\cos^2 135° + \cos^2\theta_y + \cos^2 60° = 1$$

$$\therefore \cos^2\theta_y = 0.25 \rightarrow \cos\theta_y = \pm 0.5$$

observe that θ_y is obtuse, $\therefore \theta_y = 120°$.

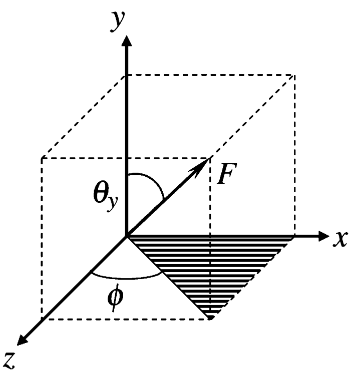
Determine the force F

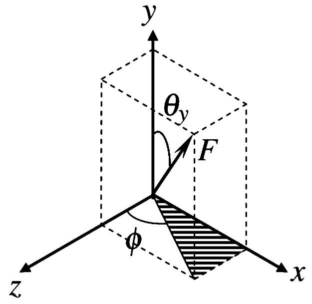
$$\begin{aligned}\mathbf{F} =& 1000\cos 135°\mathbf{i} + 1000\cos 120°\mathbf{j}\\ &+ 1000\cos 60°\mathbf{k}\\ =& -707\text{ N }\mathbf{i} - 500\text{ N }\mathbf{j} + 500\text{ N }\mathbf{k}\end{aligned}$$

4.3 Second angle method

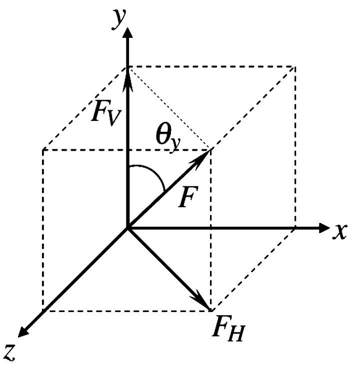
This method is used for solving the type of problem as shown below whereby;

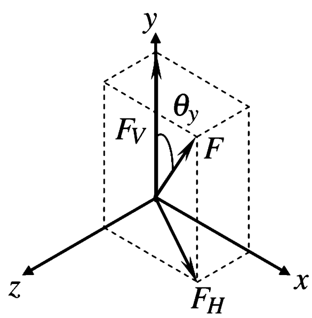
- one angle between the force and an axis is known/given (usually it's θ_y angle between the force and the y axis).
- one angle on a plane is known/given (usually the $x-z$ plane).





The force F shown above is first resolved into two components; one vertically, along the y axis, F_V and the other horizontally, on the $x-z$ plane, F_H as shown below.



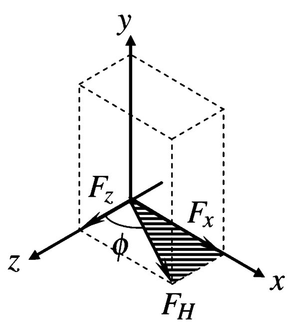


Forces F_V and F_H are obtained as;

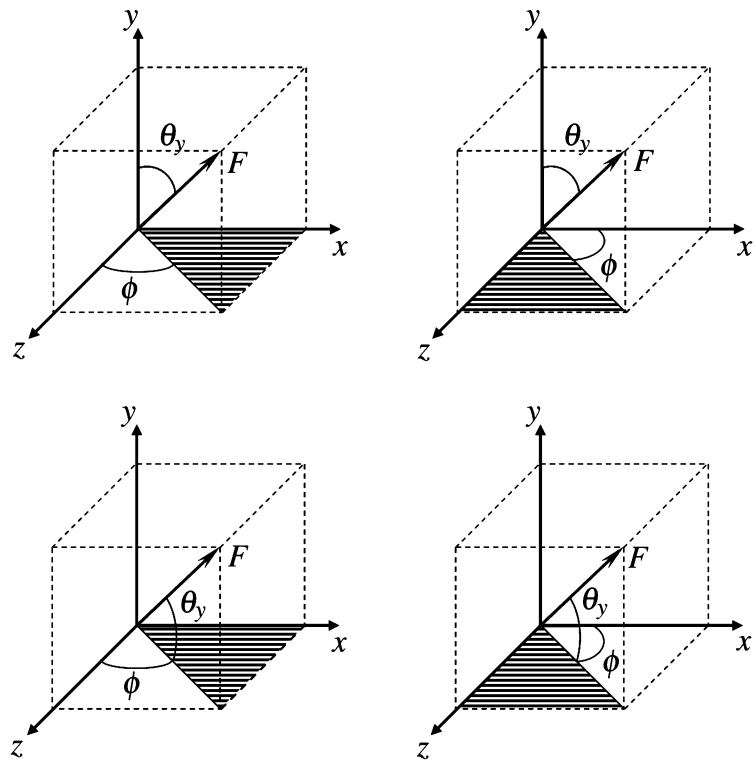
$$F_V = F\cos\theta_y = F_y$$
$$F_H = F\sin\theta_y$$

Component F_H is then resolved into two components; along the x and z axes as illustrated in the following figure giving

$$F_y = F_V = F\cos\theta_y$$
$$F_H = F\sin\theta_y$$
$$F_x = F_H\sin\phi = F\sin\theta_y\sin\phi$$
$$F_z = F_H\cos\phi = F\sin\theta_y\cos\phi$$



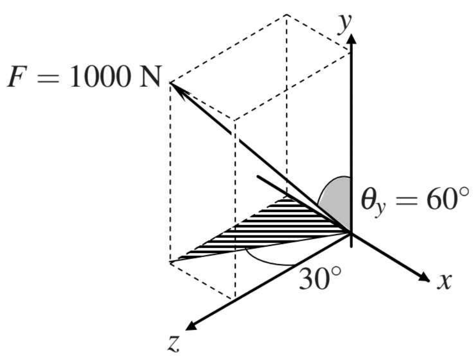
For the choice of angle, it depends largely on the information given from the problem diagram. In general there are four types of possible arrangements as follows;



■ **Example 4.2**

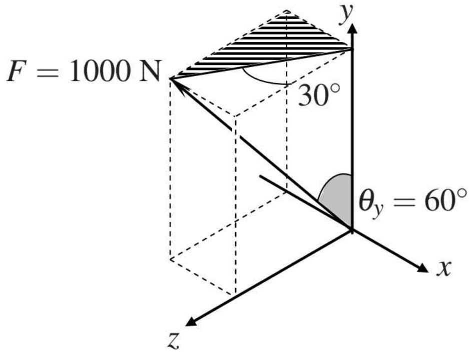
Determine the 1000 N force in terms of **i**, **j** and **k**.

Solution:



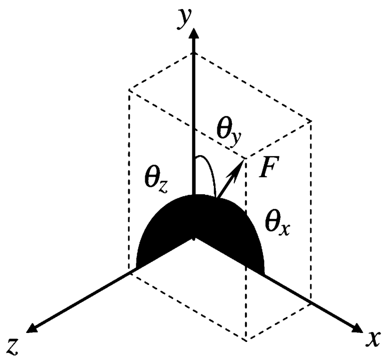
$$F_y = F_V = 1000\cos 60^\circ = 500 \text{ N}$$
$$F_H = 1000\sin 60^\circ$$
$$F_x = -F_H \sin 30^\circ = -1000\sin 60^\circ \sin 30^\circ$$
$$= -433 \text{ N}$$
$$F_z = F_H \cos 30^\circ = 1000\sin 60^\circ \cos 30^\circ = 750 \text{ N}$$
$$\therefore \vec{F}_1 = -433 \text{ N}\mathbf{i} + 500 \text{ N}\mathbf{j} + 750 \text{ N}\mathbf{k}$$

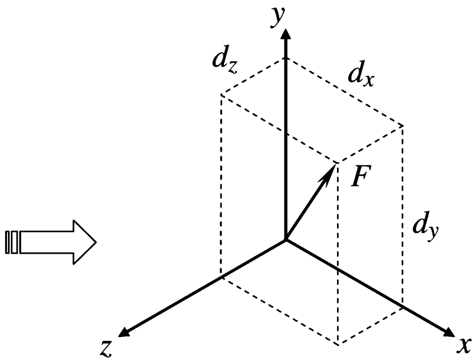
Note:
The configuration shown on the right will produce the same result.



4.4 Coordinate method

This method is suitable for problems with known/given coordinates or dimensions, i.e;





From $\mathbf{F} = F\lambda$

$$\text{thus } \mathbf{F} = F\cos\theta_x\mathbf{i} + F\cos\theta_y\mathbf{j} + F\cos\theta_z\mathbf{k}$$

whereby

$$\cos\theta_x = \lambda_x = \frac{d_x}{d} \qquad F_x = F\frac{d_x}{d}$$

$$\cos\theta_y = \lambda_y = \frac{d_y}{d} \qquad F_y = F\frac{d_y}{d}$$

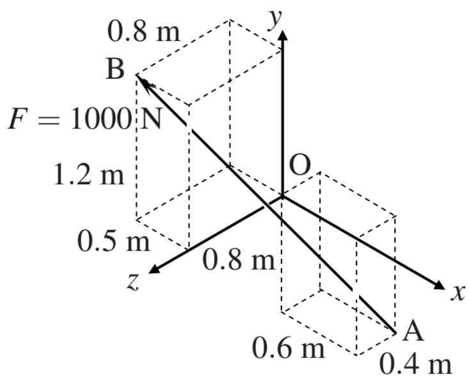
$$\cos\theta_z = \lambda_z = \frac{d_z}{d} \qquad F_z = F\frac{d_z}{d}$$

with $d = \sqrt{d_x^2 + d_y^2 + d_z^2}$.

$$\therefore \mathbf{F} = F\frac{d_x}{d}\mathbf{i} + F\frac{d_y}{d}\mathbf{j} + F\frac{d_z}{d}\mathbf{k} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

■ Example 4.3

Determine the 1000 N force in terms of **i**, **j** and **k**.



Solution:
From the figure,

$$
\begin{aligned}
d_x &= -1.1 \text{ m} \\
d_y &= 2.0 \text{ m} \\
d_z &= 1.2 \text{ m} \\
\therefore d &= \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{(-1.1)^2 + (2.0)^2 + (1.2)^2} = 2.58 \text{ m}
\end{aligned}
$$

Values of d_x, d_y and d_z can also be determined using the following method:

The direction of the force is from point A to B. Thus;

$$
\begin{array}{llll}
\text{coordinate of point A:} & (-0.5, & +1.2, & +0.8) \\
\text{coordinate of point B:} & (+0.6, & -0.8, & -0.4)\ - \\
\hline
 & (\underbrace{-1.1}_{d_x}, & \underbrace{2.0}_{d_y}, & \underbrace{1.2}_{d_z})
\end{array}
$$

then, $d = \sqrt{(-1.1)^2 + (2.0)^2 + (1.2)^2} = 2.58$ m

Therefore,

$$
\begin{aligned}
\vec{F} &= F\frac{d_x}{d}\mathbf{i} + F\frac{d_y}{d}\mathbf{j} + F\frac{d_z}{d}\mathbf{k} \\
&= 1000\left(\frac{-1.1}{2.58}\right)\mathbf{i} + 1000\left(\frac{2.0}{2.58}\right)\mathbf{j} + 1000\left(\frac{1.2}{2.58}\right)\mathbf{k} \\
&= -426 \text{ N } \mathbf{i} + 775 \text{ N } \mathbf{j} + 465 \text{ N } \mathbf{k}
\end{aligned}
$$

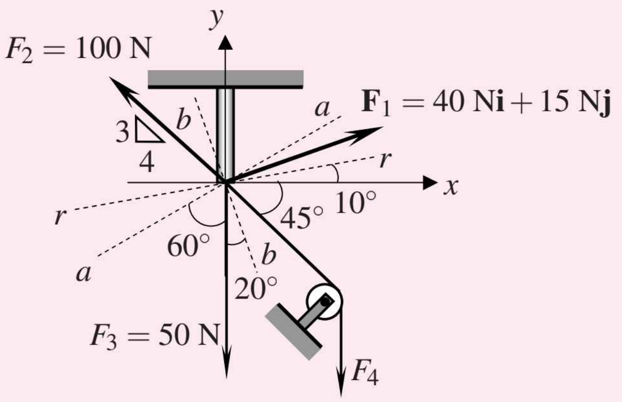
4.5 Example questions

Exercise 4.1

If the resultant R for F_1, F_2, F_3 and F_4 acts along r-r, find

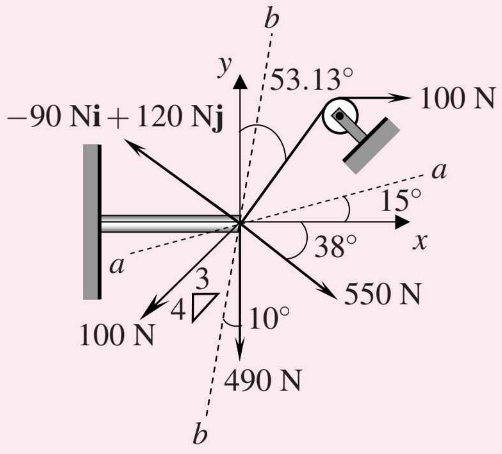
a. magnitude of F_4.
b. the resultant R.
c. the components of F_2 along a-a and b-b.

Answer: (a) $F_4 = 38.5$ N, (b) $R = 12.96$ N, (c) $F_{a-a} = 55.5$ N and $F_{b-b} = 93.4$ N



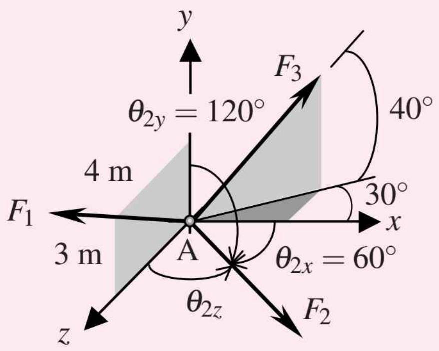
Exercise 4.2

Determine the resultant R for the five forces shown and find the two components if R is resolved along a-a and b-b.
Answer: $R = 814.2$ N, $\theta = 63.5°$,
$R_{a-a} = 534.3$ (15° ∡) and
$R_{b-b} = 880.3$ (10° ∡)



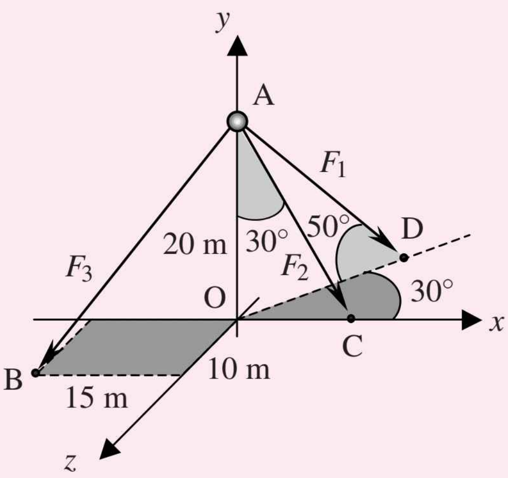
Exercise 4.3

Three forces $F_1 = 100$ N, F_2, and $F_3 = 120$N act on point A. The component of F_2 along the x-axis (F_{2x}) is positive. If the component of the resultant along the y-axis (R_y) equals 50 N, determine F_2, R_x and R_z. Calculate the magnitude of the resultant, R.
Answer: $F_2 = 174.27$ N, $R_x = 166.74$ N, $R_z = 157.26$ N and $R = 234.6$ N



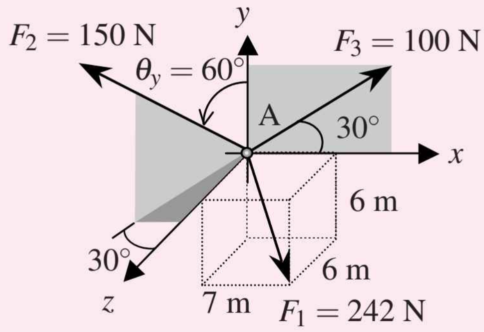
Exercise 4.4

Three concurrent forces $F_1 = 800$ N, $F_2 = 500$ N and F_3 each act at points D, C and B all located on the x–z plane. If the resultant R of the three forces acts on the x–y plane, determine the vectors of F_3 and the resultant R.
Answer:
$\mathbf{F}_3 = F_3(-0.557\mathbf{i} - 0.743\mathbf{j} + 0.371\mathbf{k})$,
$F_3 = 693$ N,
$R = 1591.1$ N ($\theta = 78.79°$ ∡)



Exercise 4.5

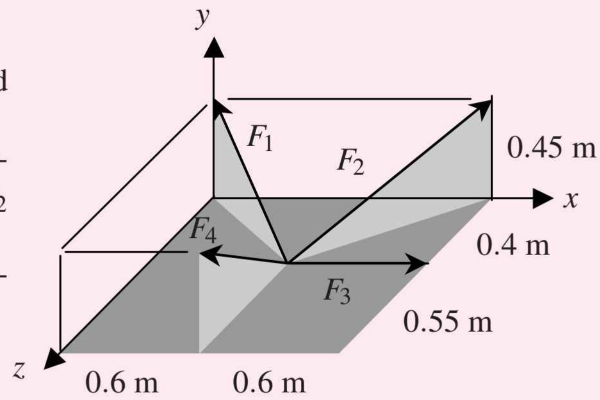
Particle A is acted upon by three forces; F_1, F_2 and F_3. Determine the resultant, R.



Exercise 4.6

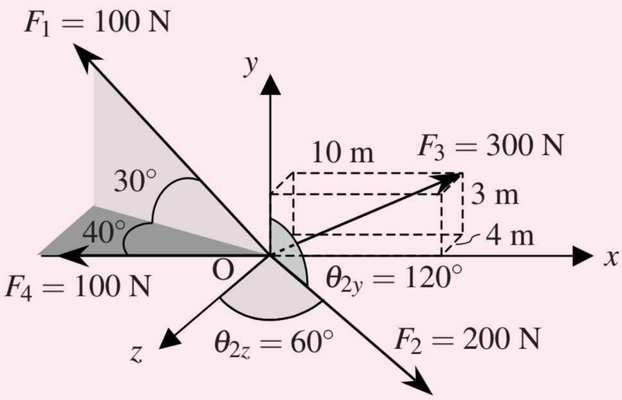
Given $F_1 = 510$ N, $F_3 = 100$ N and $F_4 = 645$ N, determine

a. F_2, if component along the x-axis of the resultant of F_1 and F_2 equals 90 N,
b. the resultant of the whole system.



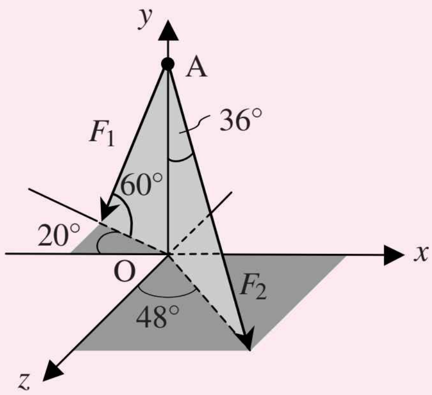
Exercise 4.7

Four forces F_1, F_2, F_3 and F_4 act on point O as shown. Determine the resultant of the system in terms $R = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$.



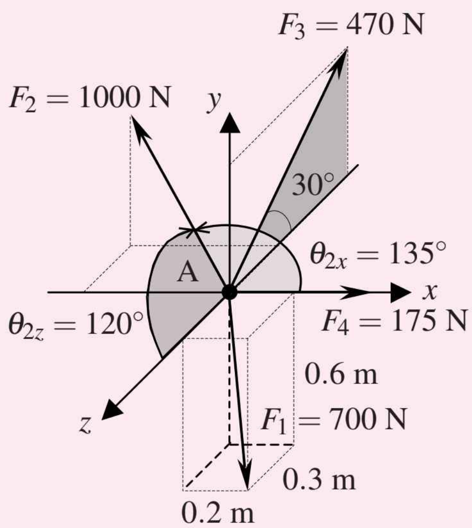
Exercise 4.8

- Given $F_1 = 200$ N and $F_2 = 250$ N. Determine the resultant, R.
- Determine the magnitude of F_1 and F_2 if the resultant R of the two forces acts on the y-z plane with a magnitude of 1000 N.
- Given $F_1 = 200$ N and the resultant, R for the two forces acts on the y–z plane. Determine R and the magnitude of F_2.



Exercise 4.9

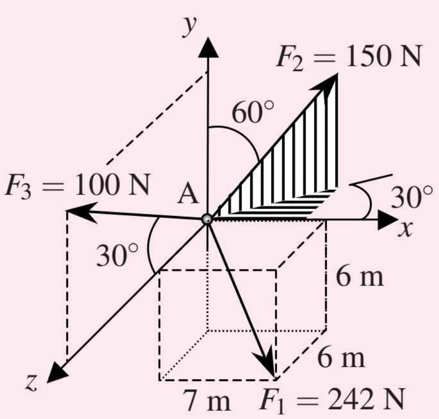
Determine the resultant R of the system.



Exercise 4.10

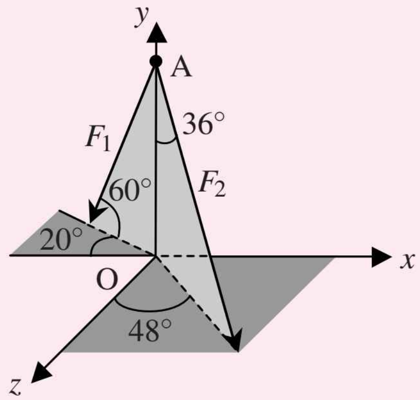
Three forces F_1, F_2 and F_3 act on particle A as shown. Determine

a. The resultant R in terms of $R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$.
b. The angle between the resultant R and the x-axis, θ_{Rx}.
c. The force F_4, if the resultant of F_1, F_2, F_3 and F_4 is zero.



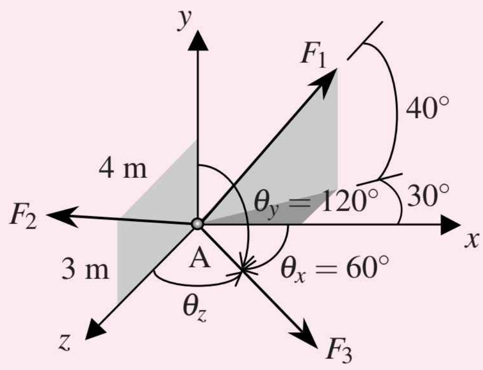
Exercise 4.11

Resultant $R = 1000$ N of the two forces, F_1 and F_2 pass through the z-axis. Determine the magnitudes of both forces.



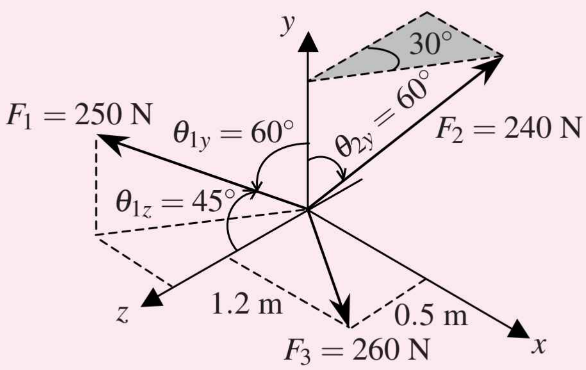
Exercise 4.12

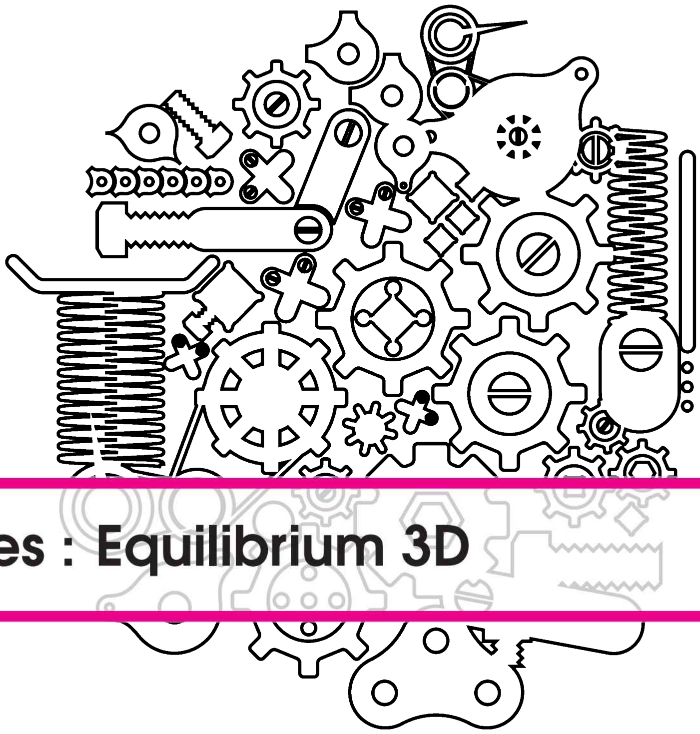
Given $F_1 = 510$ N, $F_2 = 100$ N and $F_3 = 645$ N, determine the resultant R.



Exercise 4.13

Determine the resultant of the three forces shown in terms of **i**, **j** and **k**.





5 — Particles : Equilibrium 3D

5.1 Three dimensional equilibrium analysis

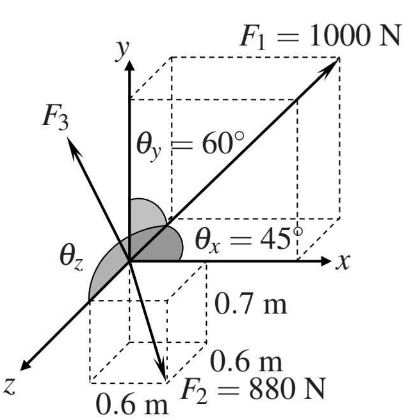
For three dimensional equilibrium of particle, the particle is in equilibrium (static) when;

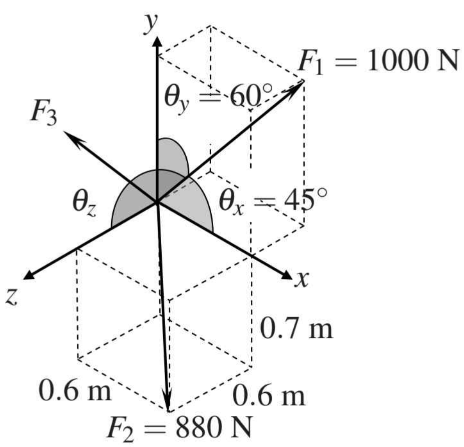
$$\text{Equilibrium} \quad \mathbf{R} = \sum \mathbf{F} = 0$$
$$\therefore \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} = 0$$

By separating into components (or axes), gives

$$\mathbf{i} \text{ component:} \quad R_x = \sum F_x = 0$$
$$\mathbf{j} \text{ component:} \quad R_y = \sum F_y = 0$$
$$\mathbf{k} \text{ component:} \quad R_z = \sum F_z = 0$$

■ **Example 5.1** Determine $\mathbf{F}_3$ if the system shown is in equilibrium.





Solution:

$\mathbf{F}_1$ (using first angle method):

From $\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$

$$\therefore \cos^2 45^\circ + \cos^2 60^\circ + \cos^2\theta_z = 1$$
$$\cos^2\theta_z = 0.25$$
$$\cos\theta_z = \pm 0.5$$

observe that θ_z is obstuse, $\therefore\ \theta_z = 120^\circ$

$$\begin{aligned}\mathbf{F}_1 &= F_1\cos\theta_x\mathbf{i} + F_1\cos\theta_y\mathbf{j} + F_1\cos\theta_z\mathbf{k}\\ &= 1000\cos 45^\circ\mathbf{i} + 1000\cos 60^\circ\mathbf{j} + 1000\cos 120^\circ\mathbf{k}\\ &= 707\text{ N }\mathbf{i} + 500\text{ N }\mathbf{j} - 500\text{ N }\mathbf{k}\end{aligned}$$

$\mathbf{F}_2$ (using coordinate method):

$$d_x = 0.6\text{ m}$$
$$d_y = -0.7\text{ m}$$
$$d_z = 0.6\text{ m}$$
$$\therefore d = \sqrt{(0.6)^2 + (-0.7)^2 + (0.6)^2} = 1.1\text{ m}$$

$$\begin{aligned}\mathbf{F}_2 &= F_2\frac{d_x}{d}\mathbf{i} + F_2\frac{d_y}{d}\mathbf{j} + F_2\frac{d_z}{d}\mathbf{k}\\ &= 880\left(\frac{0.6}{1.1}\right)\mathbf{i} + 880\left(\frac{-0.7}{1.1}\right)\mathbf{j} + 880\left(\frac{0.6}{1.1}\right)\mathbf{k}\\ &= 480\text{ N }\mathbf{i} - 560\text{ N }\mathbf{j} + 480\text{ N }\mathbf{k}\end{aligned}$$

$\mathbf{F}_3$ (unknown):

$$\mathbf{F}_3 = F_{3x}\mathbf{i} + F_{3y}\mathbf{j} + F_{3z}\mathbf{k}$$

For equilibrium, $\sum F_x\mathbf{i} + \sum F_y\mathbf{j} + \sum F_z\mathbf{k} = 0;\quad \therefore \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$

	i component	**j** component	**k** component
$\mathbf{F}_1$	+707 N	+500 N	-500 N
$\mathbf{F}_2$	+480 N	-560 N	+480 N
$\mathbf{F}_3$	$+F_{3x}$	$+F_{3y}$	$+F_{3z}$
R	0	0	0

Solving for each component gives;

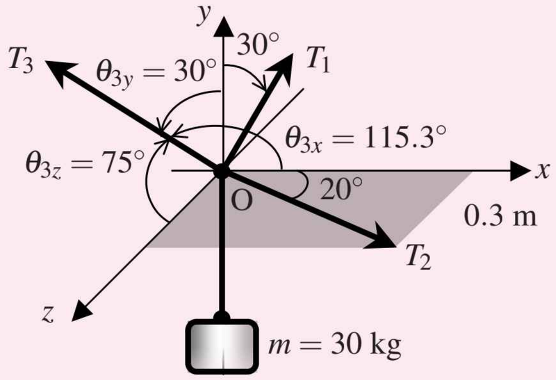
i component:	$707 + 480 + F_{3x} = 0$	$\rightarrow F_{3x} = -1187$ N
j component:	$500 - 560 + F_{3y} = 0$	$\rightarrow F_{3x} = 60$ N
k component:	$-500 + 480 + F_{3x} = 0$	$\rightarrow F_{3x} = 20$ N

$$\therefore \mathbf{F}_3 = -1187 \text{ N } \mathbf{i} + 60 \text{ N } \mathbf{j} + 20 \text{ N } \mathbf{k}$$

5.2 Example questions

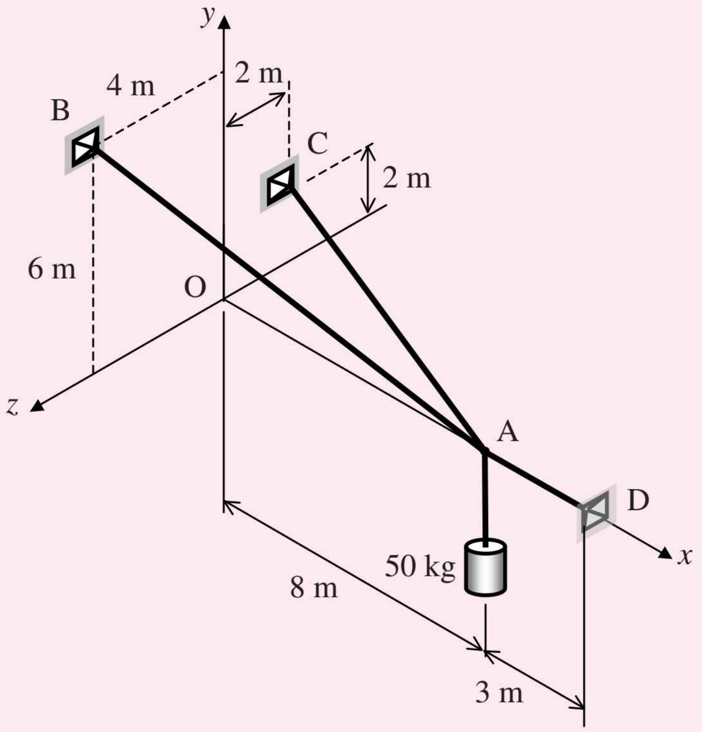
Exercise 5.1

A mass $m = 30$ kg is supported by three forces. Determine T_1, T_2 and T_3 to maintain equilibrium. Force T_1 is in y-z plane.



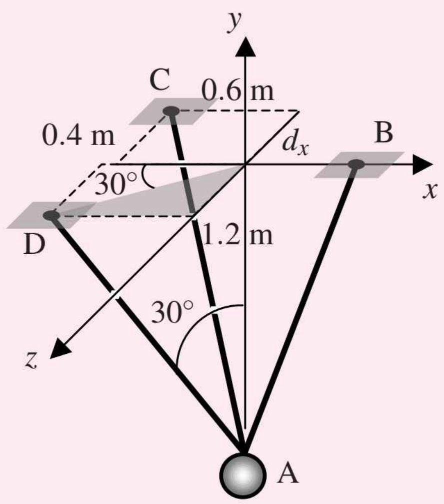
Exercise 5.2

Determine the required tension in cables AB, AC and AD to hold the 50 kg mass in equilibrium.



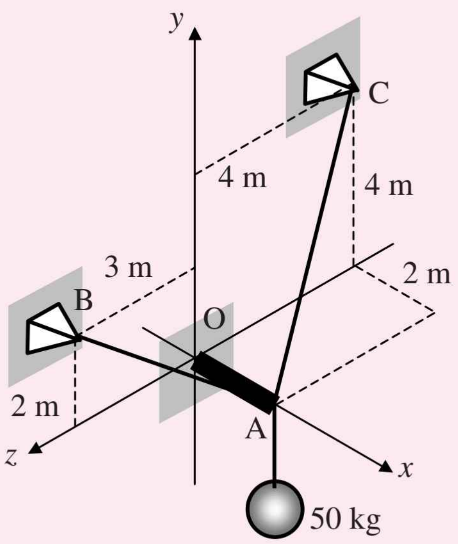
Exercise 5.3

The 40 kg mass at A is supported by cables AB, AC and AD. If the tension in cable AC, $T_{AC} = 140$ N, determine the tension in cables AB and AD, and the length d_x.
Answer: $T_{AB} = 186.1$ N, $T_{AD} = 160$ N and $d_x = 1.16$ m,



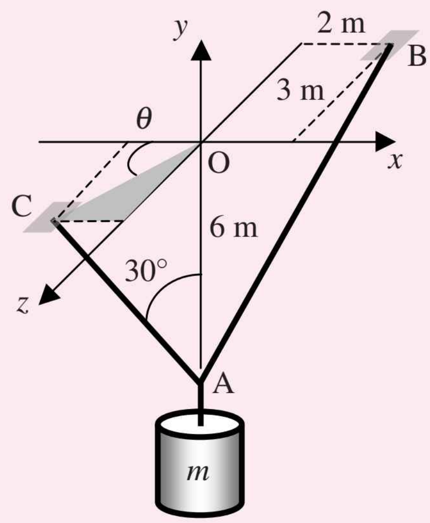
Exercise 5.4

A 50 kg mass is supported by the structure OA and cables AB and AC as shown. Determine the forces in the structure and the cables. The system is in equilibrium.

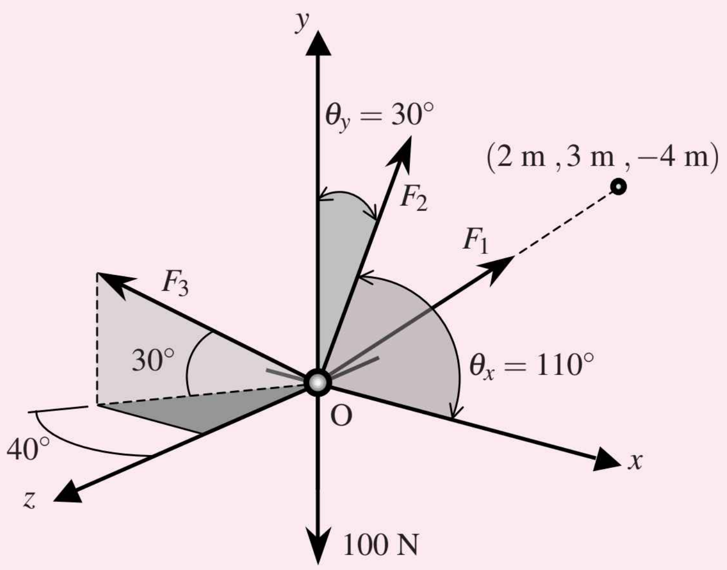


Exercise 5.5

The point A lies on the y-axis and points B and C are situated on the x-z plane. The system is in equilibrium and the tension in cable AB is 7 kN. Determine the angle θ, the tension in cable AC and the mass m.

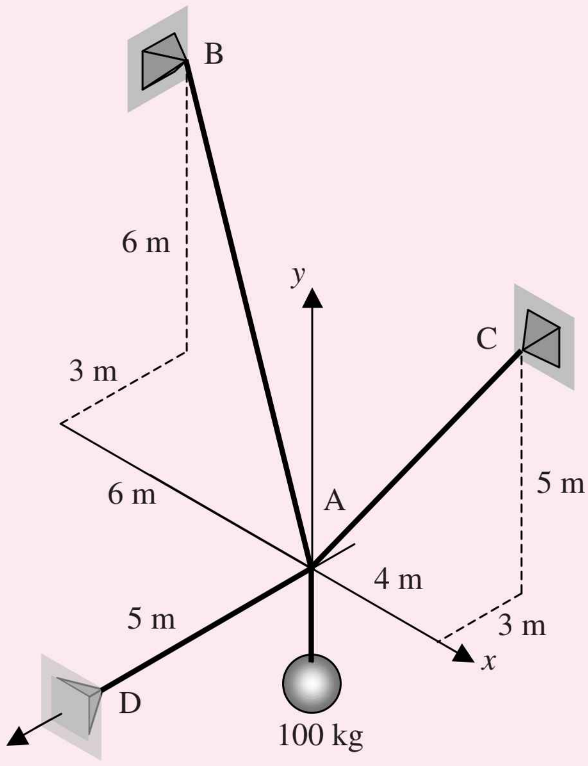


Exercise 5.6
Determine the forces F_1, F_2, and F_3 so that the system is in equilibrium.



Exercise 5.7

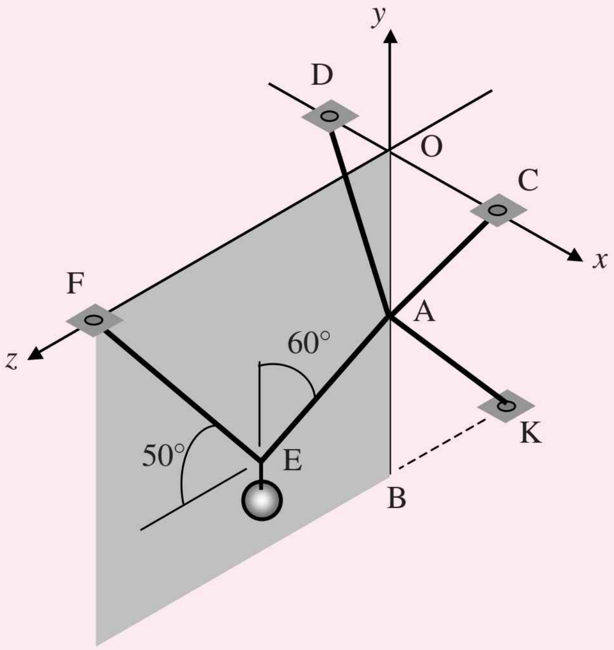
The system is in equilibrium. Determine the tension in cables AB, AC and AD to support the 100 kg mass.



Exercise 5.8

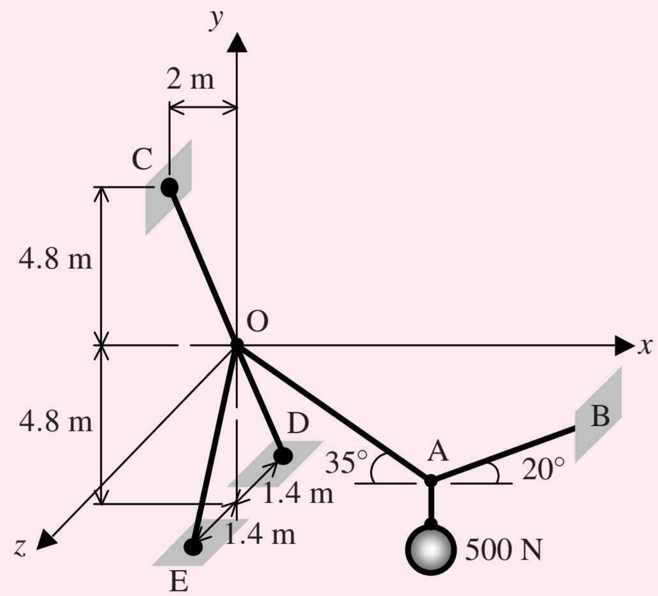
The system is in equilibrium. A 100 kg load is suspended at E. Determine the tension in all cables.

Given:
OA = AB = 1.2 m
BK = 0.8 m
OC = 0.9 m
OD = 0.3 m



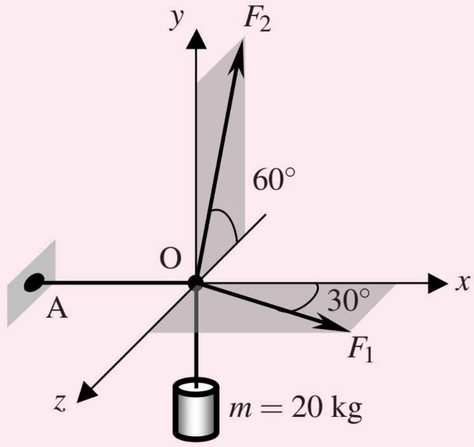
Exercise 5.9

The 500 N weight at A is supported by a five cable system as shown. Determine the tension in every cable. Points C, O, A and B are situated on the same x–y plane while points O, D and E are situated on the same y–z plane.



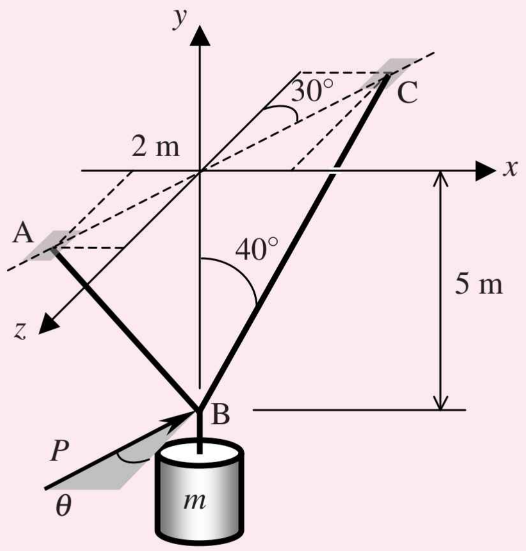
Exercise 5.10

The system in the figure is in equilibrium. Determine the tension in cable OA, and the forces F_1 and F_2.



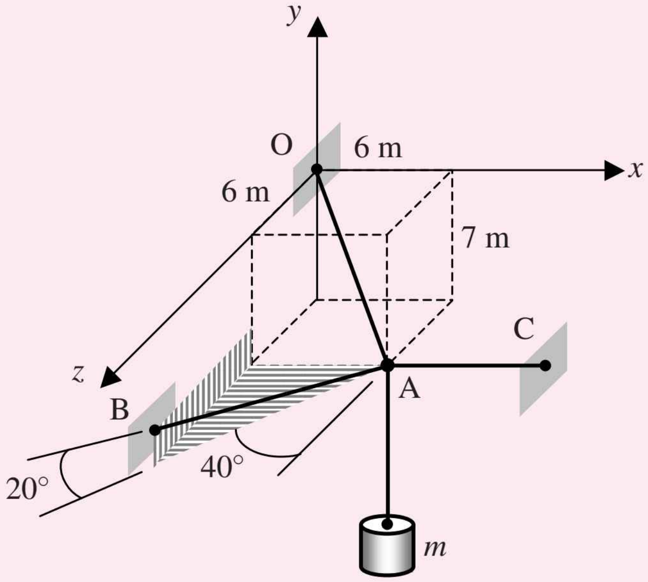
Exercise 5.11

The mass m is supported by cables AB and BC, and a force P as shown. If the tension in both cables AB and BC equal 10 kN, determine the force P, angle θ and mass m to maintain equilibrium. The force P is parallel to the x-z plane.



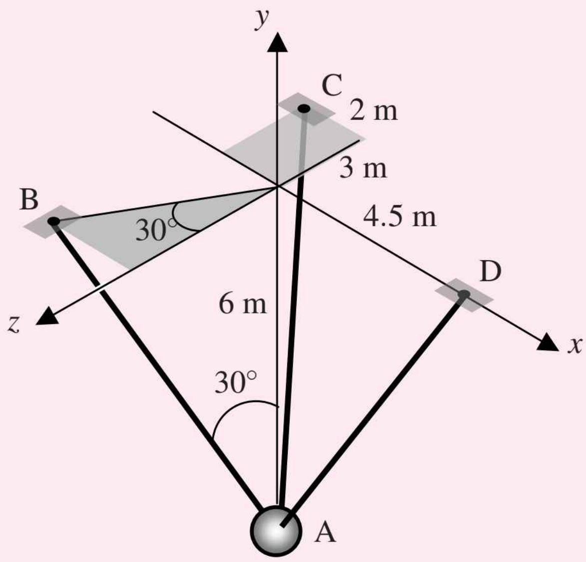
Exercise 5.12

The mass m is supported at the position shown by cables AO, AB and AC. If the tension in cable AO is 350 N, determine the tension in cables AC and AB, and mass m in kg.



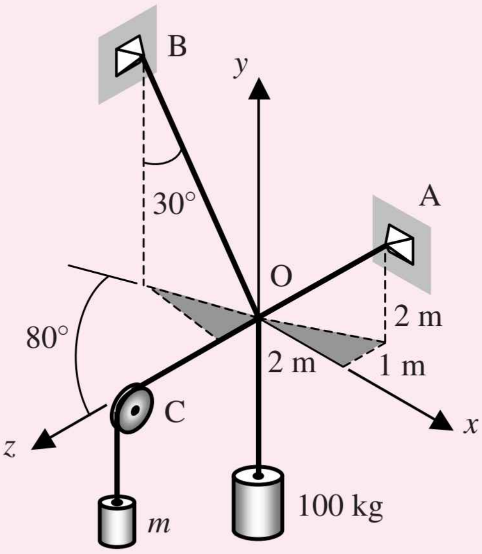
Exercise 5.13

The system is in equilibrium. If $m_A = 100$ kg determine the tension in cables AB, AC and AD.



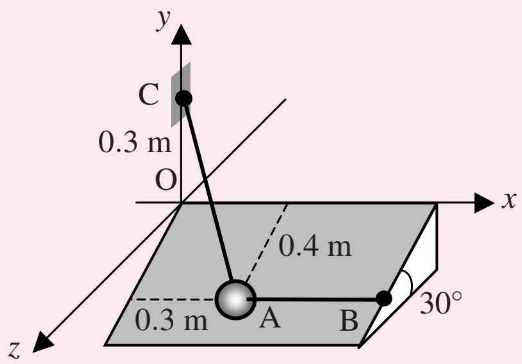
Exercise 5.14

The 100 kg mass is supported by three cables as shown. Determine the tension in each cable and the mass m for the system to maintain equilibrium.

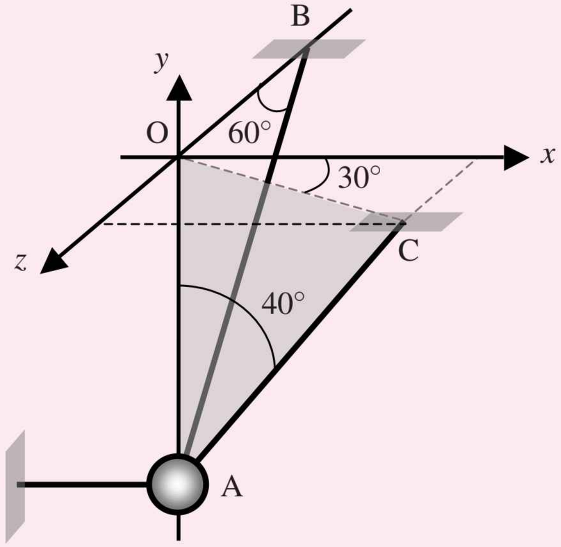


Exercise 5.15

Ball A with mass of 10 kg rests on a smooth 30° slope supported by cables AB and AC. Determine the tension in cables AB and AC.

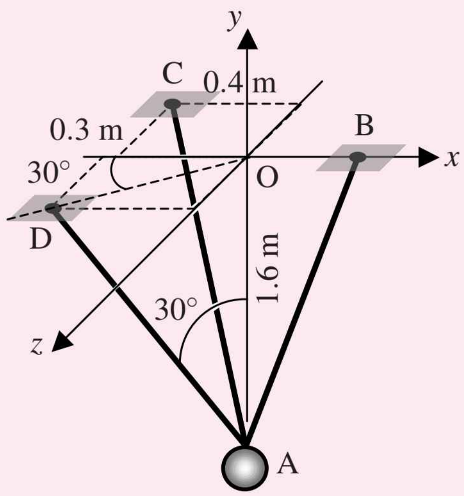


Exercise 5.16



The figure shows a 10 kg mass supported by three cables. Determine the tension in cables T_{AB}, T_{AC} and T_{AD} to maintain equilibrium.

Exercise 5.17



The mass *m* at A is supported by three cables, AB, AC and AD. Points B, C and D are situated on the *x*–*z* plane and length OB = 1.2 m. If the tension in cable AD is 750 N, determine the tension in cables AB and AC, and the mass *m* to maintain equilibrium.

6 — Rigid bodies : Moment 2D

6.1 Analysis of rigid bodies

Previously, only concurrent forces are considered in analyses when bodies are treated as particles. In reality, forces act at different points of application, so the size of the body will have to be taken into consideration. Thus the assumption that all forces are concurrent is no longer valid. Bodies considered in statics are assumed to be rigid. A rigid body is defined as a body that does not deform. Hence, each of the forces acting on a rigid body can impart a motion of translation or rotation, or both.

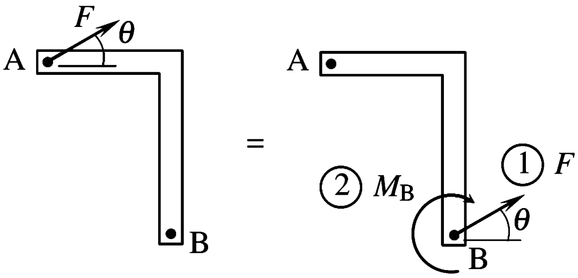
Forces exerted on rigid bodies can be categorized as;

- **External forces**
  Externally applied force or reaction force of other bodies on outer surface of the rigid body under consideration (applied through contact or joint).
- **Internal forces**
  Forces that hold particles forming a rigid body together. This internal force can only be seen at the parting location of the considered rigid body when it is being cut.

In Statics course we deal mostly with external forces. Internal forces are studied in detail typically in mechanics of solid courses.

6.2 Moment of a force about a point in 2D

Consider the problem shown in the following figure. A force F is acting on point A of a rigid body. What is the effect of this force on point B? (where B is another point on the rigid body). Point B will feel both the force and a rotation due to force F which is called a moment about a point. Since we are considering only for 2D analysis, both points A and B are on the same plane. Moment of a force about a point in 2D is a vector quantity having both magnitude and direction.

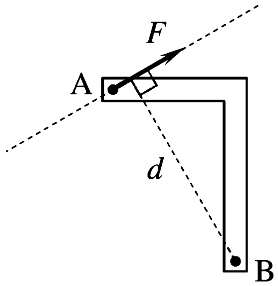


Point B will experience
① a pull by force F
② a rotation due to force F
→ this is moment !!

For example, the moment of a force F about a point B, M_B is defined by

$$M_B = F \cdot d \circlearrowright$$

where d is the perpendicular/nearest distance between the **line of action of the force** F and **point** B. Observe that the moment of force F about point B does not depend on point A, but on the line of action of the force.



Direction of a moment is referred to as clockwise ($\circlearrowright$ / CW) or counter clockwise ($\circlearrowleft$ / CCW). For the above example, the force F will generate a clockwise moment about point B.The usual sign convention is CCW +ve ($\circlearrowleft$+). However, when answering exam questions, students may choose either CW or CCW as the positive sign but need to be clearly stated in the answer.

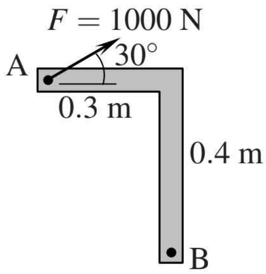
Concept 6.1 — Varignon's Theorem. The moment of a force about a point is equivalent to the summation of the moments of the force's components about that point.

Concept 6.2 — Principle of Transmissibility. The moment of a force about a point is the same as long as the force lies on its line of action.

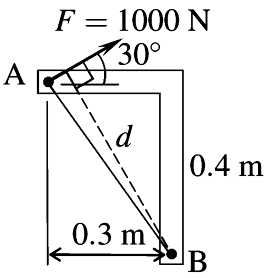
■ Example 6.1

Determine the moment of the 1000 N force about point B by:

(a) definition of moment,
(b) resolving the force into it's components.



Solution (a):



$(\circlearrowright +)\ M_B = F \cdot d$

$$AB = \sqrt{(0.3)^2 + (0.4)^2} = 0.5 \text{ m}$$

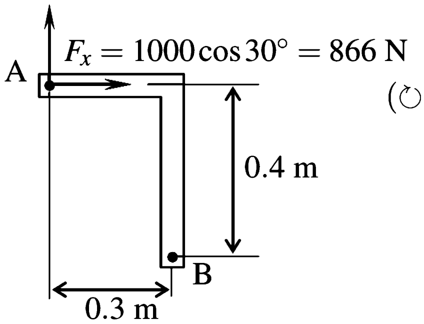
$$\angle BAF = 30^\circ + \tan^{-1}\frac{0.4}{0.3} = 83.13^\circ$$

$$\sin\angle BAF = \frac{d}{AB} \quad \Rightarrow \quad d = 0.496 \text{ m}$$

$$\therefore M_B = (1000 \text{ N})(0.496 \text{ m}) = 496 \text{ Nm}(\circlearrowright)$$

Solution (b):

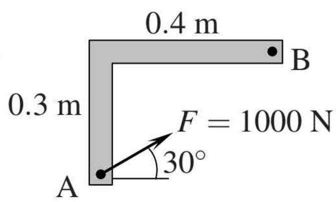
$F_y = 1000 \sin 30^\circ = 500$ N



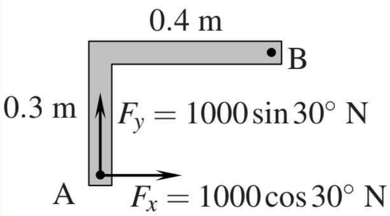
$$\begin{aligned}(\circlearrowright +)\ M_B &= F \cdot d \\ &= +(1000 \sin 30^\circ)(0.3 \text{ m}) + (1000 \cos 30^\circ)(0.4 \text{ m}) \\ &= 150 + 345.8 \\ &= 496 \text{ N.m}(\circlearrowright)\end{aligned}$$

■ Example 6.2

Determine the moment of the 1000 N force about point B by resolving the force into its components and use the result to find the shortest distance between the line of force and point B.

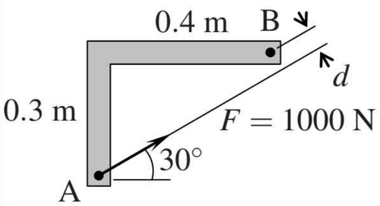


Solution:



$$\begin{aligned}(\circlearrowright +)\ M_B &= F \cdot d \\ &= +(1000 \sin 30^\circ)(0.4 \text{ m}) - (1000 \cos 30^\circ)(0.3 \text{ m}) \\ &= 200 - 259.8 = -59.8 \text{ Nm}(\circlearrowright)\end{aligned}$$

$$\therefore M_B = 59.8 \text{ Nm}(\circlearrowleft)$$



$(\circlearrowleft +)\ M_B = F \cdot d$

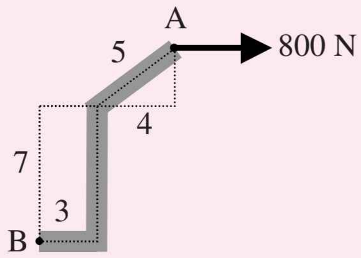
$$59.8 \text{ Nm} = 1000\text{N} \cdot d$$

$$\therefore d = 0.0598 \text{ m} = 0.06 \text{ m}$$

6.3 Example questions

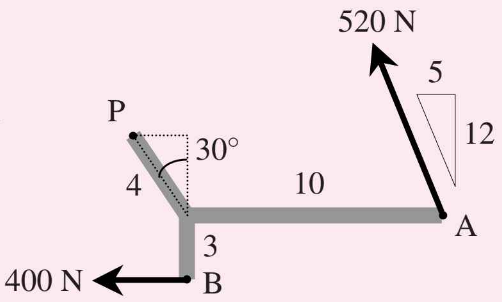
Exercise 6.1

Determine the moment of the 800 N force about point B. All dimensions in m.
Answer: $M_B = 8000$ Nm (↻)



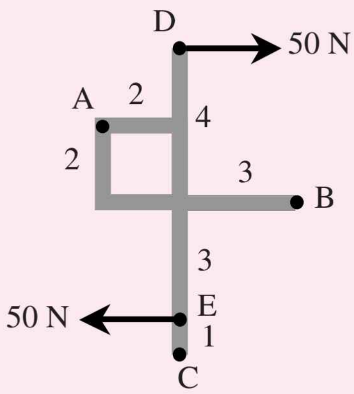
Exercise 6.2

Determine the moment of the forces about point P. All dimensions in m.
Answer: $M_P = 2484$ Nm (↺)



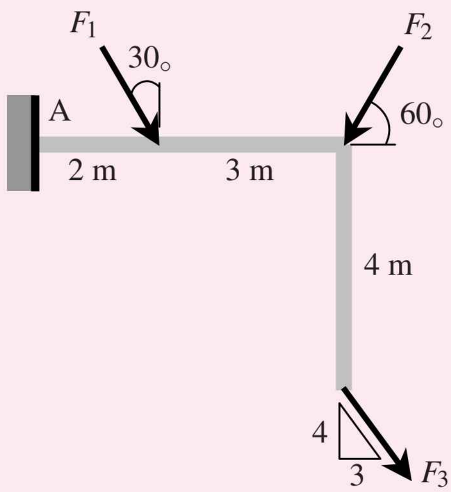
Exercise 6.3

Determine the moment of the two 50 N forces about points A, B and C. All dimensions in m.



Exercise 6.4

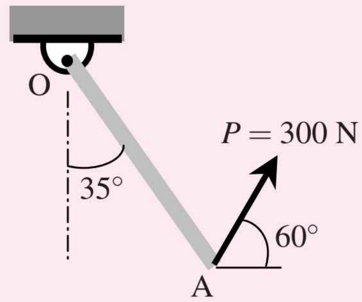
Determine the magnitude of F_3 if the moment of all the forces about point A equals 4800 Nm (↻). Given $F_1 = 300$ N and $F_2 = 400$ N.
Answer: $F_3 = 1592.7$ N



Exercise 6.5

The arm OA is pin jointed at O where OA = 200 mm. Determine the moment of the force P about point O using :

a. the definition of moment.
b. Varignon's theorem (rectangular components of forces).
c. parallel and perpendicular components of OA.

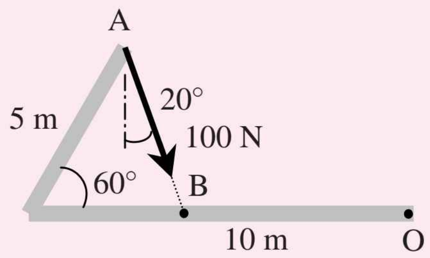


Exercise 6.6

Determine the moment of the force 100 N about point O using :

a. components of the force at A.
b. components of the force at B, by first transmitting the force at A to B.

Determine the shortest distance between the line of action of the 100 N force and point O.



7 — Rigid bodies : Moment 3D

7.1 Moment of a force about a point in 3D

Moment about a point in 3D is given by the cross product of position vector $\mathbf{r}$ and force vector $\mathbf{F}$ as

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where;

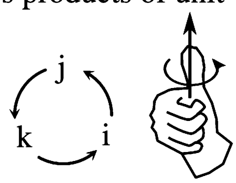
$$\begin{aligned}
\mathbf{r} &= \text{position vector} \\
&\quad \textit{start} \rightarrow \text{ the point where moment is taken} \\
&\quad \textit{end} \rightarrow \text{ the point where the force acts} \\
\mathbf{r} &= r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k} \\
\text{and} \quad \mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}
\end{aligned}$$

therefore moment about a point $\mathbf{M}$ is;

$$\mathbf{M} = (r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})$$

For solving the moment equation, the concept of vector cross product is used. In 3D analysis, $\mathbf{r}$ and $\mathbf{F}$ are resolved into unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components corresponding to x, y and z axis, respectively. These axes are always 90° to each other and 0° with respect to themselves. Since $\sin 90° = 1$ and $\sin 0° = 0$, the cross products of unit vectors are;

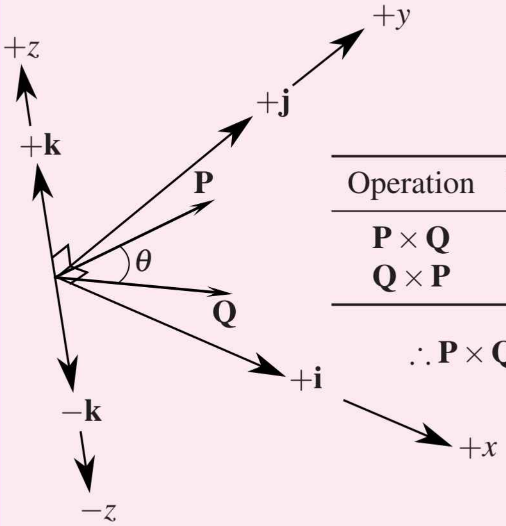
$$\begin{array}{lll}
\mathbf{i} \times \mathbf{i} = 0 & \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\
\mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{j} = 0 & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\
\mathbf{i} \times \mathbf{k} = -\mathbf{j} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{k} = 0
\end{array}$$



Therefore, moment can be obtained as;

$$\begin{aligned}
\mathbf{M} &= (r_xF_y)\mathbf{k} - (r_xF_z)\mathbf{j} - (r_yF_x)\mathbf{k} + (r_yF_z)\mathbf{i} + (r_zF_x)\mathbf{j} - (r_zF_y)\mathbf{i} \\
&= (r_yF_z - r_zF_y)\mathbf{i} + (r_zF_x - r_xF_z)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k}
\end{aligned}$$

Concept 7.1 Vector cross product

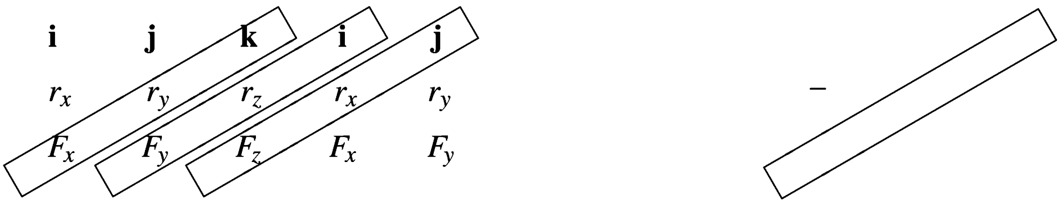


Operation	Magnitude	Direction
$\mathbf{P}\times\mathbf{Q}$	$PQ\sin\theta$	$-\mathbf{k}$
$\mathbf{Q}\times\mathbf{P}$	$PQ\sin\theta$	$+\mathbf{k}$

$\therefore \mathbf{P}\times\mathbf{Q} \neq \mathbf{Q}\times\mathbf{P}$

Right hand rule

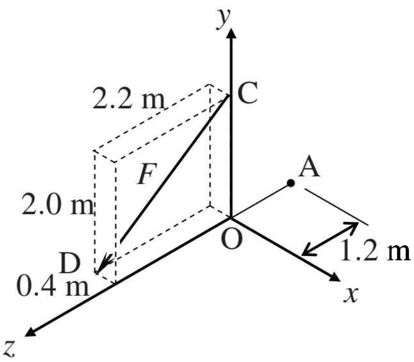
The equation can also be solved using determinant;



$$\therefore \mathbf{M} = (r_yF_z)\mathbf{i} + (r_zF_x)\mathbf{j} + (r_xF_y)\mathbf{k} - (r_zF_y)\mathbf{i} - (r_xF_z)\mathbf{j} - (r_yF_x)\mathbf{k}$$
$$= (r_yF_z - r_zF_y)\mathbf{i} + (r_zF_x - r_xF_z)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k}$$

▪ Example 7.1

(a) Determine the moment of force $F = 3000$ N about point A.
(b) Determine the shortest distance between point A and line of action of the force.



Solution (a):

$$\mathbf{M}_A = \mathbf{r}_{AC} \times \mathbf{F}_{CD}$$

$$\mathbf{F}_{\mathrm{CD}} \Rightarrow \quad \mathbf{F}_{\mathrm{CD}} = F_{\mathrm{CD}}\lambda_{\mathrm{CD}}$$
$$= F_{\mathrm{CD}}\left(\frac{d_x}{d}\mathbf{i} + \frac{d_y}{d}\mathbf{j} + \frac{d_z}{d}\mathbf{k}\right)$$
$$= 3000\left(\frac{-0.4}{3}\mathbf{i} + \frac{-2.0}{3}\mathbf{j} + \frac{2.2}{3}\mathbf{k}\right)$$
$$= -400 \text{ N } \mathbf{i} - 2000 \text{ N } \mathbf{j} + 2200 \text{ N } \mathbf{k}$$

from C to D :
$d_x = -0.4$ m
$d_y = -2.0$ m
$d_z = 2.2$ m
$\therefore d = 3.0$ m

$$\mathbf{r} \Rightarrow \quad \mathbf{r}_{\mathrm{AC}} = 2.0 \text{ m } \mathbf{j} + 1.2 \text{ m } \mathbf{k}$$
$$\mathbf{r}_{\mathrm{AD}} = -0.4 \text{ m } \mathbf{i} + 3.4 \text{ m } \mathbf{k}$$

Vector **r** can be chosen from either $\mathbf{r}_{\mathrm{AC}}$ or $\mathbf{r}_{\mathrm{AD}}$ because both points C and D are on the force vector line.

For the purpose of this exercise, lets choose $\mathbf{r}_{\mathrm{AC}}$;

$$\mathbf{M}_{\mathrm{A}} = \mathbf{r}_{\mathrm{AC}} \times \mathbf{F}_{\mathrm{CD}}$$
$$= (2.0 \text{ m } \mathbf{j} + 1.2 \text{ m } \mathbf{k}) \times (-400 \text{ N } \mathbf{i} - 2000 \text{ N } \mathbf{j} + 2200 \text{ N } \mathbf{k})$$
$$= (2.0 \text{ m})(-400 \text{ N})(-\mathbf{k}) + (2.0 \text{ m})(2200 \text{ N})(\mathbf{i})$$
$$+ (1.2 \text{ m})(-400 \text{ N})(\mathbf{j}) + (1.2 \text{ m})(-2000 \text{ N})(-\mathbf{i})$$
$$= 800 \text{ Nm } \mathbf{k} + 4400 \text{ Nm } \mathbf{i} - 480 \text{ Nm } \mathbf{j} + 2400 \text{ Nm } \mathbf{i}$$
$$= 6800 \text{ Nm } \mathbf{i} - 480 \text{ Nm } \mathbf{j} + 800 \text{ Nm } \mathbf{k}$$

the result will be the same if $\mathbf{r}_{\mathrm{AD}}$ is chosen.

Solution (b):

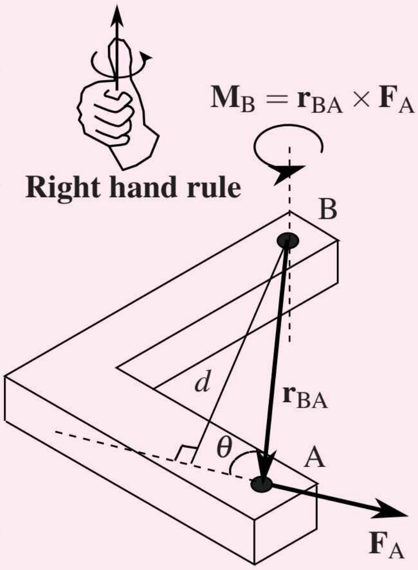
$$M_A = \sqrt{M_x^2 + M_y^2 + M_z^2} = \sqrt{6800^2 + (-480)^2 + 800^2} = 6863.7 \text{ Nm}$$

from $M_A = Fd \Rightarrow d = \dfrac{M_A}{F} = \dfrac{6863.7}{3000} = 2.288$ m

Why $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ not $\mathbf{M} = \mathbf{F} \times \mathbf{r}$?

Consider our previous 2D moment about a point problem represented in a 3D form. Here, both points A and B lies on the same plane. Thus the moment produced by force $\mathbf{F}_\mathrm{A}$ at point B is a 2D moment on this plane. The magnitude of this moment is given by;

$$\begin{aligned} M_\mathrm{B} &= F_\mathrm{A}.d \\ &= F_\mathrm{A}.(r_\mathrm{BA} \sin\theta) \\ &= |\mathbf{r}_\mathrm{BA} \times \mathbf{F}_\mathrm{A}| \end{aligned}$$



The direction of this moment should also follows the right hand rule. Therefore, this proves why the moment equation must be $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ and not $\mathbf{M} = \mathbf{F} \times \mathbf{r}$.

7.2 Moment of a force about an axis in 3D

Moment of a force about an axis is defined as

$$M_\mathrm{axis} = \boldsymbol{\lambda}_\mathrm{axis} \cdot \mathbf{M}_\mathrm{point} \quad \text{i.e.} \quad M_\mathrm{AB} = \boldsymbol{\lambda}_\mathrm{AB} \cdot (\mathbf{M}_\mathrm{A} \text{ or } \mathbf{M}_\mathrm{B})$$

where;

M_AB = moment of force **F** about axis AB

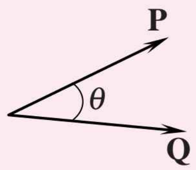
$\boldsymbol{\lambda}_\mathrm{AB}$ = unit vector from A to B

$\mathbf{M}_\mathrm{A}$ or $\mathbf{M}_\mathrm{B}$ = moment of force **F** about point A or point B

Moment of a force about an axis is a scalar quantity due to the vector dot product operation. In a nutshell;

Moment of a force about an axis (scalar)	=	Unit vector of the axis (vector)	dot	Moment of the force about any point on that axis (vector)

Concept 7.2 **Vector dot product**



Operation	Magnitude	Direction
$\mathbf{P}\cdot\mathbf{Q}$	$PQ\cos\theta$	not associated
$\mathbf{Q}\cdot\mathbf{P}$	$PQ\cos\theta$	not associated

$$\therefore \mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P}$$

In 3D analysis, vectors are resolved into unit vectors **i**, **j** and **k** components corresponding to x, y and z axis, respectively. These axes are always 90° to each other and 0° with respect to themselves. Since $\cos 90° = 0$ and $\cos 0° = 1$, the dot products of unit vectors are;

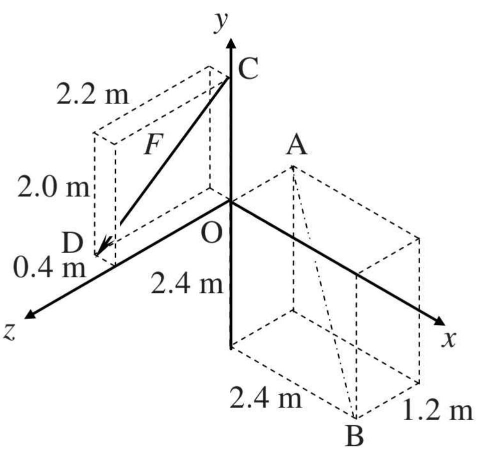
$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}=1 & \mathbf{j}\cdot\mathbf{i}=0 & \mathbf{k}\cdot\mathbf{i}=0 \\ \mathbf{i}\cdot\mathbf{j}=0 & \mathbf{j}\cdot\mathbf{j}=1 & \mathbf{k}\cdot\mathbf{j}=0 \\ \mathbf{i}\cdot\mathbf{k}=0 & \mathbf{j}\cdot\mathbf{k}=0 & \mathbf{k}\cdot\mathbf{k}=1 \end{array}$$

Vector dot product can also be used for following applications;

- To determine the angle between two vectors.
- To determine the perpendicular / 90° / nearest distance between two vectors (the line of action).

▪ Example 7.2

Determine the moment of force vector **F** with magnitude 3000 N about the axis AB.



Solution:

$$M_{AB} = \boldsymbol{\lambda}_{AB}\cdot(\mathbf{M}_A \quad \text{or} \quad \mathbf{M}_B)$$

$$\begin{aligned} M_{AB} &= \boldsymbol{\lambda}_{AB}\cdot\mathbf{M}_A = \boldsymbol{\lambda}_{AB}\cdot\overbrace{(\mathbf{r}_{AC}\times\mathbf{F}_{CD})}^{\mathbf{M}_A} = \boldsymbol{\lambda}_{AB}\cdot\overbrace{(\mathbf{r}_{AD}\times\mathbf{F}_{CD})}^{\mathbf{M}_A} \\ &= \boldsymbol{\lambda}_{AB}\cdot\mathbf{M}_B = \boldsymbol{\lambda}_{AB}\cdot\overbrace{(\mathbf{r}_{BC}\times\mathbf{F}_{CD})}^{\mathbf{M}_B} = \boldsymbol{\lambda}_{AB}\cdot\overbrace{(\mathbf{r}_{BD}\times\mathbf{F}_{CD})}^{\mathbf{M}_B} \end{aligned}$$

Observe that $\boldsymbol{\lambda}_{AB}$ and $\mathbf{F}_{CD}$ are common to all equations, the difference is in the position vector $\mathbf{r}$. Unless otherwise stated, it is advisable to choose the 'simplest' $\mathbf{r}$.

$$
\begin{aligned}
\boldsymbol{\lambda}_{AB} \Rightarrow \quad \boldsymbol{\lambda}_{AB} &= \lambda_{ABx}\mathbf{i} + \lambda_{ABy}\mathbf{j} + \lambda_{ABz}\mathbf{k} \\
&= \frac{d_x}{d}\mathbf{i} + \frac{d_y}{d}\mathbf{j} + \frac{d_z}{d}\mathbf{k} \\
&= \frac{2.4}{3.6}\mathbf{i} + \frac{-2.4}{3.6}\mathbf{j} + \frac{1.2}{3.6}\mathbf{k} \\
&= \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}
\end{aligned}
$$

from A to B:
$d_x = 2.4$ m $d_y = -2.4$ m $d_z = 1.2$ m $d = 3.6$ m

$$
\begin{aligned}
\mathbf{F}_{CD} \Rightarrow \quad & F_{CD}\boldsymbol{\lambda}_{CD} \\
&= F_{CD}\left(\frac{d_x}{d}\mathbf{i} + \frac{d_y}{d}\mathbf{j} + \frac{d_z}{d}\mathbf{k}\right) \\
&= 3000\left(\frac{-0.4}{3}\mathbf{i} + \frac{-2.0}{3}\mathbf{j} + \frac{2.2}{3}\mathbf{k}\right) \\
&= -400\text{ N }\mathbf{i} - 2000\text{ N }\mathbf{j} + 2200\text{ N }\mathbf{k}
\end{aligned}
$$

from C to D:
$d_x = -0.4$ m $d_y = -2.0$ m $d_z = 2.2$ m $d = 3.0$ m

$$
\begin{aligned}
\mathbf{r}_{AC} &= 2.0\text{ m }\mathbf{j} + 1.2\text{ m }\mathbf{k} \\
\mathbf{r}_{AD} &= -0.4\text{ m }\mathbf{i} + 3.4\text{ m }\mathbf{k} \\
\mathbf{r}_{BC} &= -2.4\text{ m }\mathbf{i} + 4.4\text{ m }\mathbf{j} \\
\mathbf{r}_{BD} &= -2.8\text{ m }\mathbf{i} + 2.4\text{ m }\mathbf{j} + 2.2\text{ m }\mathbf{k}
\end{aligned}
$$

Vector $\mathbf{r}$ can be chosen from either $\mathbf{r}_{AC}$, $\mathbf{r}_{AD}$ or $\mathbf{r}_{BC}$. Vector $\mathbf{r}_{BD}$ should not be chosen as it is the 'hardest' to use.

For the purpose of this exercise, lets choose $\mathbf{r}_{AC}$.

$$
\begin{aligned}
M_{AB} &= \boldsymbol{\lambda}_{AB} \cdot (\mathbf{r}_{AC} \times \mathbf{F}_{CD}) \\
&= \left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot \Big[(2.0\text{ m }\mathbf{j} + 1.2\text{ m }\mathbf{k}) \times (-400\text{ N }\mathbf{i} - 2000\text{ N }\mathbf{j} + 2200\text{ N }\mathbf{k})\Big] \\
&= \left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot \Big[(2.0\text{ m})(-400\text{ N})(-\mathbf{k}) + (2.0\text{ m})(2200\text{ N})(\mathbf{i}) \\
&\qquad + (1.2\text{ m})(-400\text{ N})(\mathbf{j}) + (1.2\text{ m})(2000\text{ N})(\text{-}\mathbf{i})\Big] \\
&= \left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot (800\text{ Nm }\mathbf{k} + 4400\text{ Nm }\mathbf{i} - 480\text{ Nm }\mathbf{j} + 2400\text{ Nm }\mathbf{i}) \\
&= \left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot (6800\text{ Nm }\mathbf{i} - 480\text{ Nm }\mathbf{j} + 800\text{ Nm }\mathbf{k}) \\
&= \left(\frac{2}{3}\right)(6800\text{ Nm}) + \left(-\frac{2}{3}\right)(-480\text{ Nm}) + \left(+\frac{1}{3}\right)(800\text{ Nm}) \\
&= \frac{13600}{3} + \frac{960}{3} + \frac{800}{3} = 5120\text{ Nm}
\end{aligned}
$$

Checking, using a different **r**, choose $\mathbf{r}_{AD}$.

$$\begin{aligned}
M_{AB} =&\boldsymbol{\lambda}_{AB} \cdot (\mathbf{r}_{AD} \times \mathbf{F}_{CD})\\
=&\left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot \Big[(-0.4 \text{ m } \mathbf{i} + 3.4 \text{ m } \mathbf{k}) \times (-400 \text{ N } \mathbf{i} - 2000 \text{ N } \mathbf{j} + 2200 \text{ N } \mathbf{k})\Big]\\
=&\left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot \Big[(-0.4 \text{ m})(-2000 \text{ N})(\mathbf{k}) + (-0.4 \text{ m})(2200 \text{ N})(-\mathbf{j})\\
&+ (3.4 \text{ m})(-400 \text{ N})(\mathbf{j}) + (3.4 \text{ m})(-2000 \text{ N})(\text{-}\mathbf{i})\Big]\\
=&\left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot (800 \text{ Nm } \mathbf{k} + 880 \text{ Nm } \mathbf{j} - 1360 \text{ Nm } \mathbf{j} + 6800 \text{ Nm } \mathbf{i})\\
=&\left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \cdot (6800 \text{ Nm } \mathbf{i} - 480 \text{ Nm } \mathbf{j} + 800 \text{ Nm } \mathbf{k})\\
=&\left(\frac{2}{3}\right)(6800 \text{ Nm}) + \left(-\frac{2}{3}\right)(-480 \text{ Nm}) + \left(+\frac{1}{3}\right)(800 \text{ Nm})\\
=&\frac{13600}{3} + \frac{960}{3} + \frac{800}{3} = 5120 \text{ Nm}
\end{aligned}$$

The (+ve) answer shows that the direction of the moment about the axis AB is from A to B. Use the right hand rule to determine the direction of rotation (with the thumb directing from A to B). The reverse is true if the answer is (–ve).

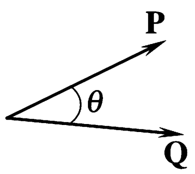
7.2.1 Angle between two vectors

Consider the angle between vectors **P** and **Q** as shown. The vectors are written in terms of its components in **i**, **j** and **k** as;

$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k} \qquad \mathbf{Q} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}$$

therefore using vector dot product gives;

$$\begin{aligned}
\mathbf{P}.\mathbf{Q} =&(P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \cdot (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})\\
=&P_xQ_x + P_yQ_y + P_zQ_z\\
=&PQ\cos\theta
\end{aligned}$$



knowing that

$$P = \sqrt{P_x^2 + P_y^2 + P_z^2}, \quad \text{and} \quad Q = \sqrt{Q_x^2 + Q_y^2 + Q_z^2}$$

thus

$$\begin{aligned}
&P_xQ_x + P_yQ_y + P_zQ_z = PQ\cos\theta\\
&\therefore \cos\theta = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{PQ}
\end{aligned}$$

It can be concluded that as long as the components of the two forces are known, the angle between then can be determined using vector dot product.

Example 7.3
Determine the angle between vectors, $\mathbf{P} = 6\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}$ and $\mathbf{Q} = -6\mathbf{i} + 33\mathbf{j} - 30\mathbf{k}$.

Solution:

$$\cos\theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ}$$

$$P_x Q_x + P_y Q_y + P_z Q_z = (6)(-6) + (6)(33) + (-7)(-30) = 372$$

$$P = \sqrt{P_x^2 + P_y^2 + P_z^2} = \sqrt{(6)^2 + (6)^2 + (-7)^2} = 11$$

$$Q = \sqrt{Q_x^2 + Q_y^2 + Q_z^2} = \sqrt{(-6)^2 + (33)^2 + (-30)^2} = 45$$

$$\cos\theta = \frac{372}{(11)(45)} = 0.7515 \quad \Rightarrow \quad \theta = 41.38^\circ$$

7.2.2 Nearest distance between two vectors

The moment about an axis and the vector dot product operation can also be used to determine the perpendicular or 90° or nearest distance between two vectors (or vectors line of action). Consider a generic vector **F**. Its component that is parallel with an axis is given by

$$F_{\text{parallel}} = \boldsymbol{\lambda}_{\text{axis}} \cdot \mathbf{F}$$

By using the Pythagoras theorem, the magnitude of the perpendicular component of **F** referring to that axis is

$$|\mathbf{F}_{\text{perpendicular}}| = \sqrt{(|\mathbf{F}|)^2 - (|\mathbf{F}_{\text{parallel}}|)^2}$$

or

$$F_{\text{perpendicular}} = \sqrt{(F)^2 - (F_{\text{parallel}})^2}$$

From the definition of moment about an axis,

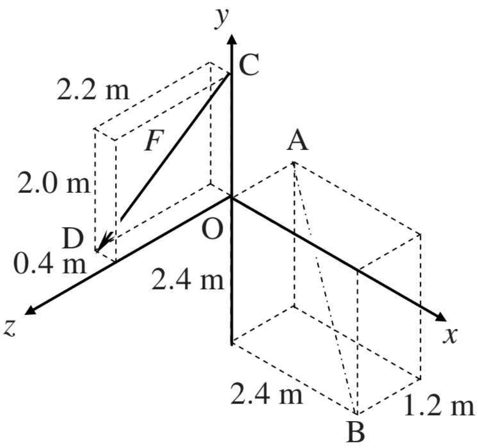
$$M_{\text{axis}} = F_{\text{perpendicular}} \cdot d$$

with d as the perpendicular distance between the line of force **F** with that particular axis. Hence;

$$d = \frac{M_{\text{axis}}}{F_{\text{perpendicular}}}$$

■ **Example 7.4**

Determine the perpendicular/shortest distance between the line of action of force $\mathbf{F} = -400\ \text{N}\ \mathbf{i} - 2000\ \text{N}\ \text{j} + 2200\ \text{N}\ \mathbf{k}$ and the line AB.



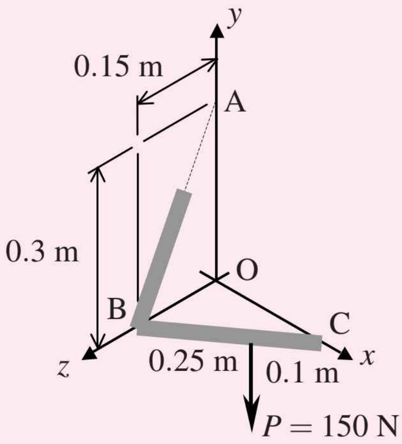
Solution:
From Example 7.2, $M_{\text{axis}} = M_{\text{AB}} = 4586.7$ Nm and $F = 3000$ N.

$$\begin{aligned}
F_{\text{parallel}} &= \boldsymbol{\lambda}_{\text{AB}} \cdot \mathbf{F} \\
&= \left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right) \cdot (-400\ \text{N}\ \mathbf{i} - 2000\ \text{N}\ \mathbf{j} + 2200\ \text{N}\ \mathbf{k}) \\
&= \left(\frac{2}{3}\right)(-400\ \text{N}) + \left(\frac{-2}{3}\right)(-2000\ \text{N}) + \left(\frac{-1}{3}\right)(2200\ \text{N}) \\
&= -\frac{1}{3}(800) + \frac{1}{3}(4000) - \frac{1}{3}(2200) = \frac{1}{3}(1000)\ \text{N} \\
F_{\text{perpendicular}} &= \sqrt{(F)^2 - (F_{\text{parallel}})^2} \\
&= \sqrt{(3000)^2 - \left(\frac{1000}{3}\right)^2} = 2981.42\ \text{N} \\
\therefore d &= \frac{M_{\text{axis}}}{F_{\text{perpendicular}}} = \frac{4586.7}{2981.42} = 1.538\ \text{m}
\end{aligned}$$

7.3 Example questions

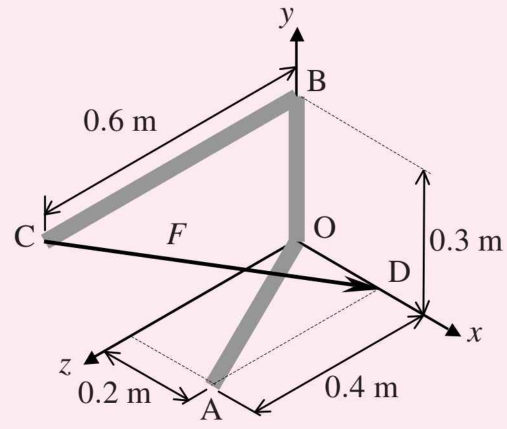
Exercise 7.1

Determine the moment of the $P = 150$ N force about the diagonal BA.
Answer: $M_{\text{BA}} = 15.2$ Nm



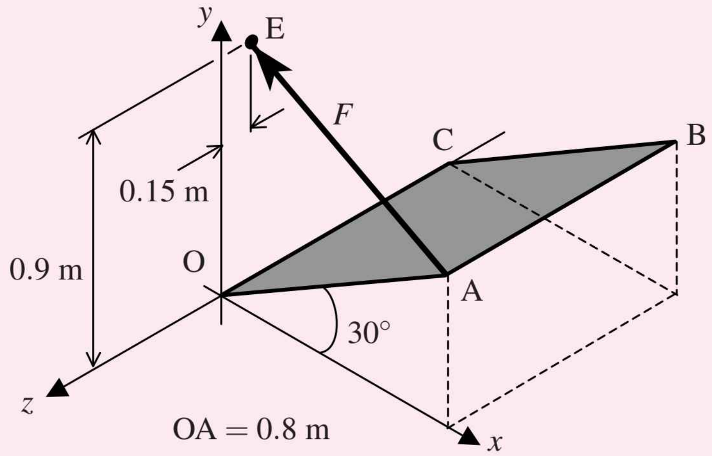
Exercise 7.2

The force $F = 700$ N acts along the line CD. Determine the moment of F about the OA axis.



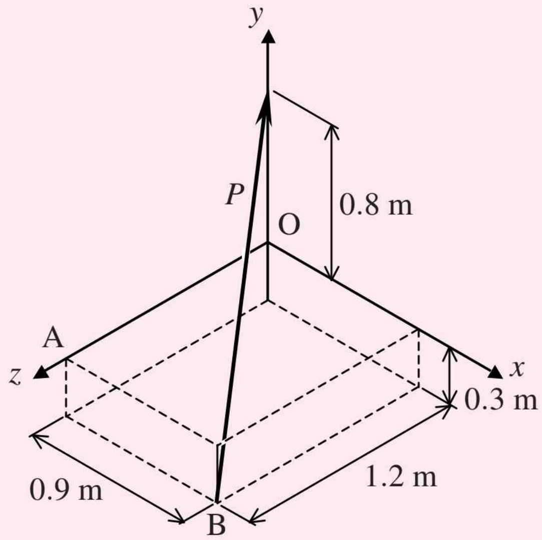
Exercise 7.3

To maintain the door OABC at the position shown, the force F at point A needs to generate a moment of –80 Nm about the OC axis. Find the magnitude of F.
Answer: $F = 111.11$ N



Exercise 7.4

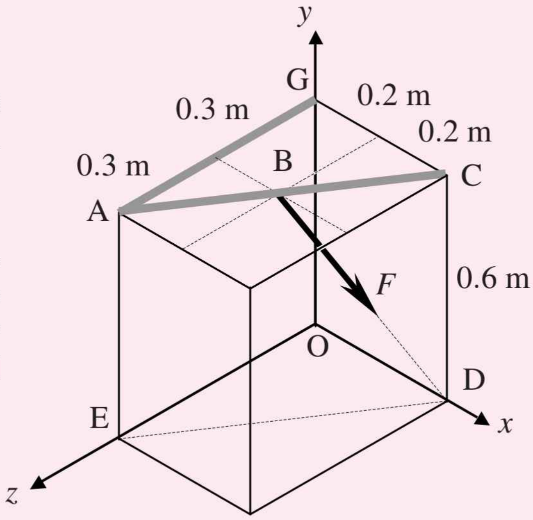
The plate is acted upon by force $P = 3640$ N as shown in the figure. Determine the moment of the force about the OA axis.



Exercise 7.5

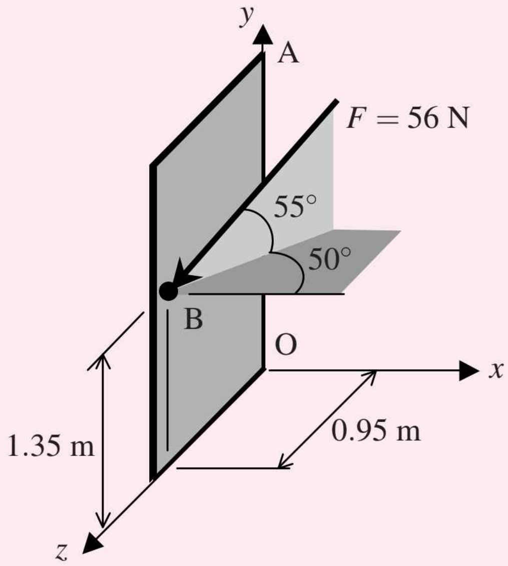
The $F = 700$ N force acts on the frame as shown. Point B is the mid point of AC. Determine:

a. the moment of F about point A.
b. the moment of F about the GA axis.
c. the frame ABC and the force F lie on the ACED plane. Explain why the moment of F about the diagonal AC equal zero.



Exercise 7.6

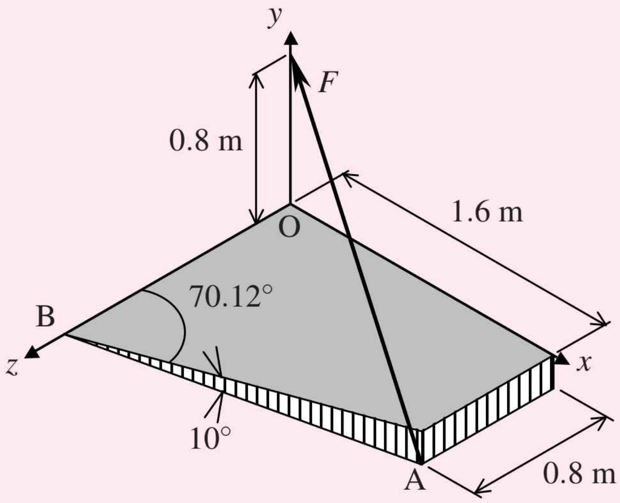
The force $F = 56$ N acts on the door knob B as shown. Determine the moment of F about O and about the OA axis.



Exercise 7.7

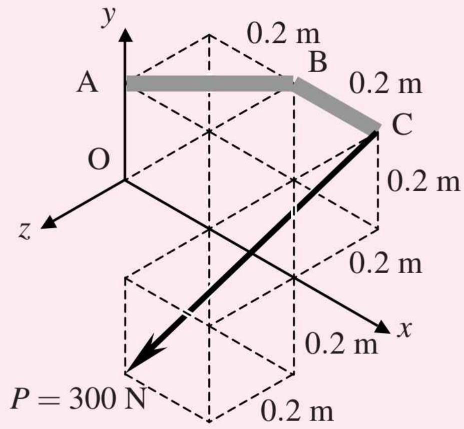
Determine:

a. The moment of force $F = 42$ N (acting at A) about point B.
b. The perpendicular distance between the line of action of the force and point B.



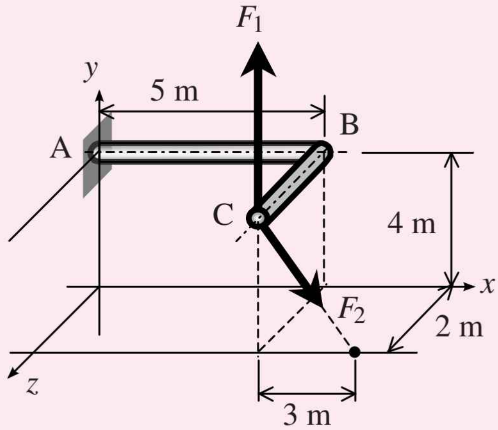
Exercise 7.8

Determine the moment of the force P about point B and about the diagonal AB.



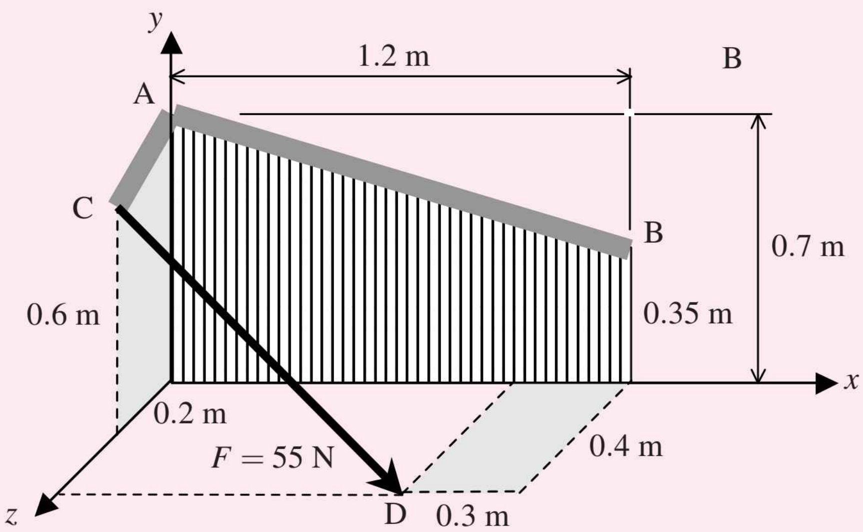
Exercise 7.9

Two forces, $F_1 = 800$ N and $F_2 = 2000$ N act at point C on the inverted L shaped frame as shown. Determine the moment about point A.
Answer:
$\mathbf{M}_A = 1600\ \text{Nm}\mathbf{i} + 2400\ \text{Nm}\mathbf{j} - 4000\ \text{Nm}\mathbf{k}$



Exercise 7.10

Determine the moment of the force F about the line AB.

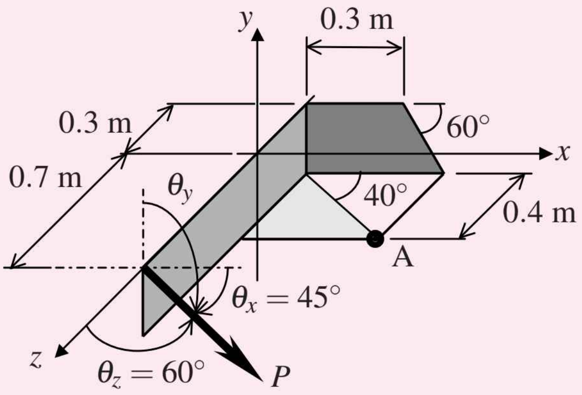


Exercise 7.11

Determine:

a. The moment of the force $P = 1000$ N about point A.

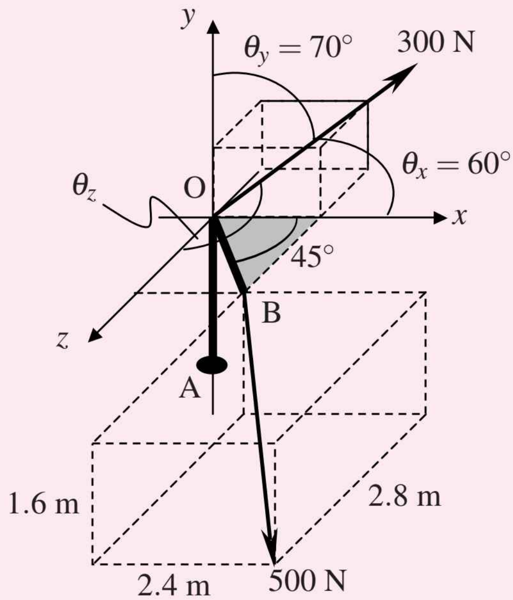
b. The nearest (perpendicular) distance between point A and the line of action of force P.



Exercise 7.12

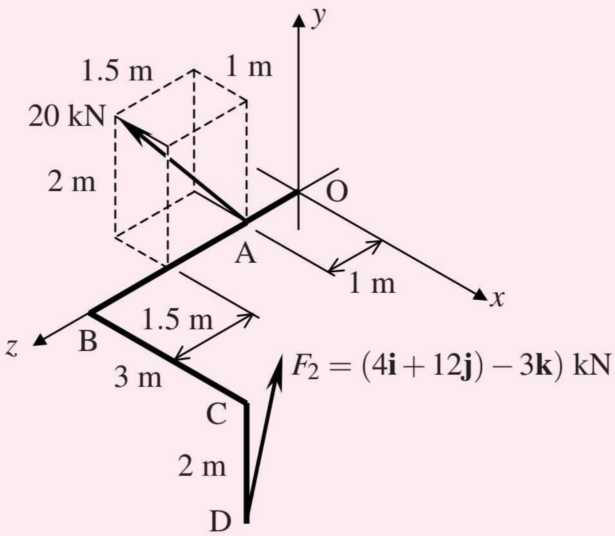
Two forces, a 300 N acts at O and a 500 N acts at B as shown. Determine the moment generated by these forces about point A.

Given OA = OB = 2 m.



Exercise 7.13

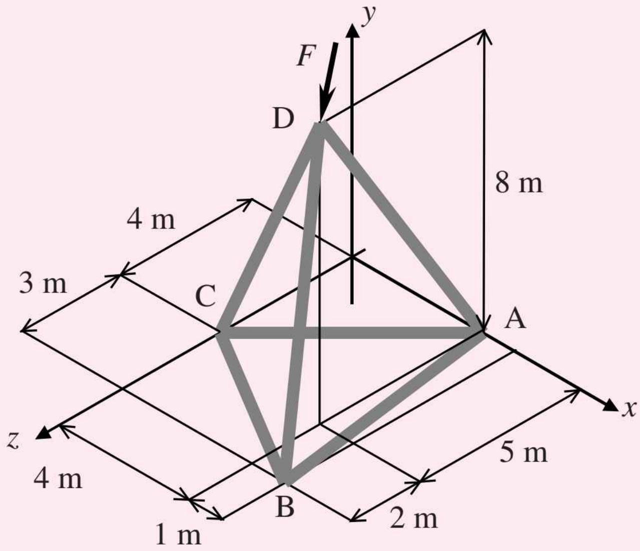
Two forces act on the bent rod as shown. Determine the moment of the forces along the diagonal OC.



Exercise 7.14

The force F acts at point D as shown in the figure. Determine the magnitude of the moment of the force about the base line CA of the tripod.

Given
$F = -20\ \text{N}\mathbf{i} - 80\ \text{N}\mathbf{j} + 50\ \text{N}\mathbf{k}$

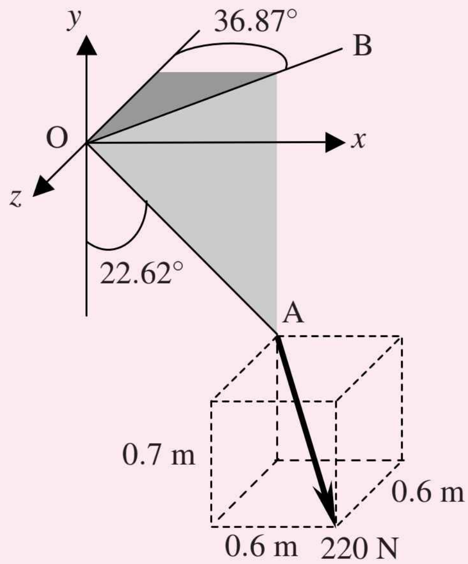


Exercise 7.15

Determine:

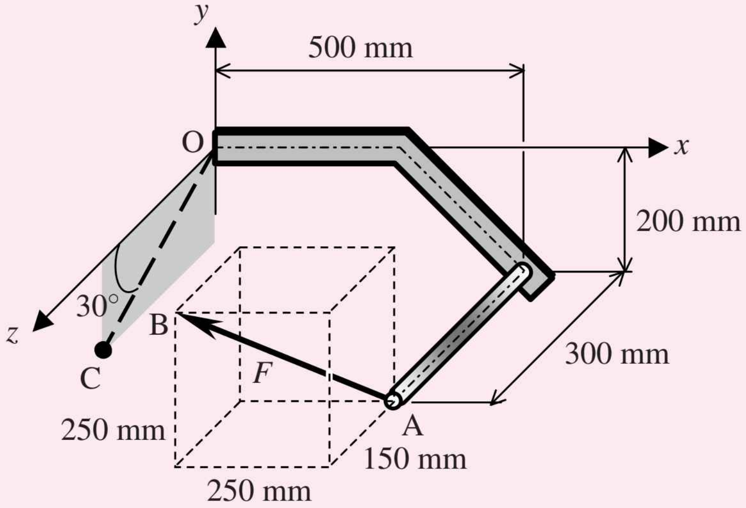
a. The moment of the 220 N force about point O.
b. The moment of the 220 N force about the diagonal OB.

Given OA = 0.13 m.

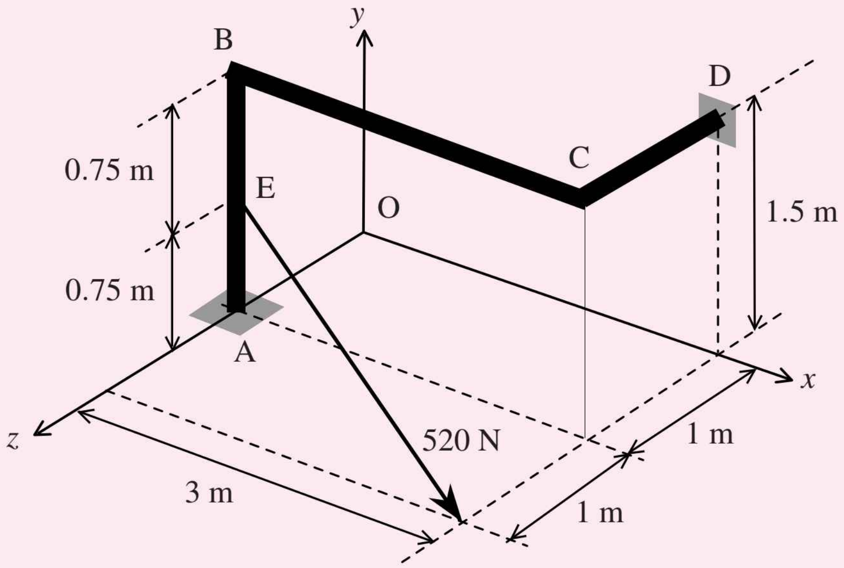


Exercise 7.16
The figure shows a force $F = 350$ N acting at A. Determine:

a. The moment of force F about point O.
b. the moment of force F about point OC axis.

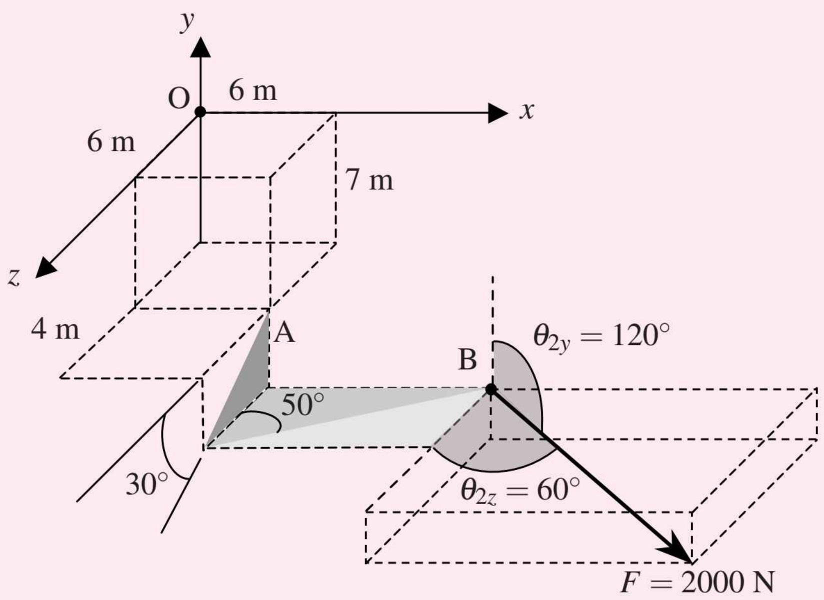


Exercise 7.17
The 520 N force is acting at point E as shown. Determine the moment of the force about the AD axis.



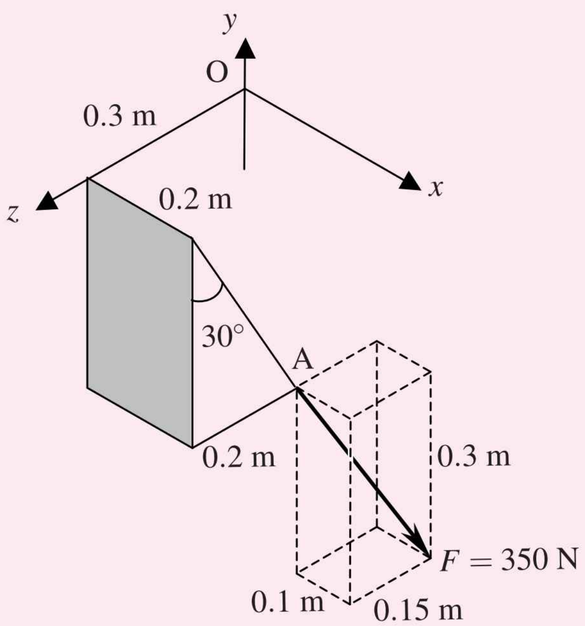
Exercise 7.18

Calculate the moment of force F about point O. Determine the perpendicular distance from the line of action of force F to point O.



Exercise 7.19

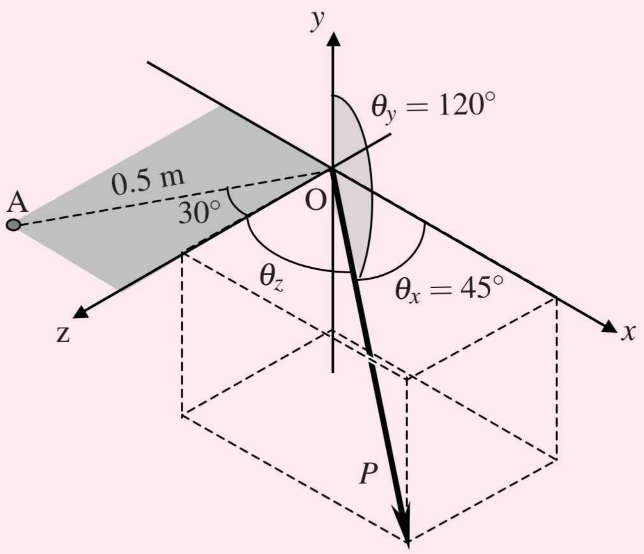
Determine the moment of the 500 N at A about point O.



Exercise 7.20

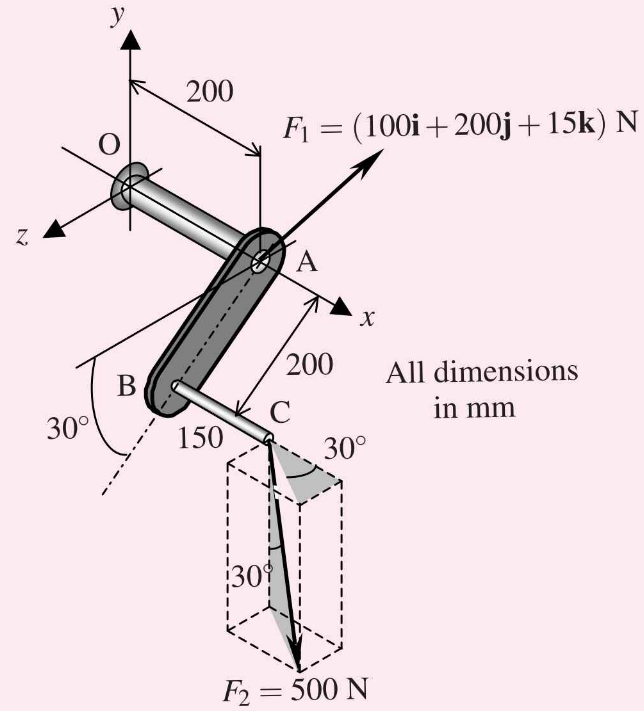
Determine:

a. The moment of the force $P = 1000$ N about point A.
b. The nearest distance between point A and the line of action of force P.



Exercise 7.21

Determine the moment of the forces shown about point O.

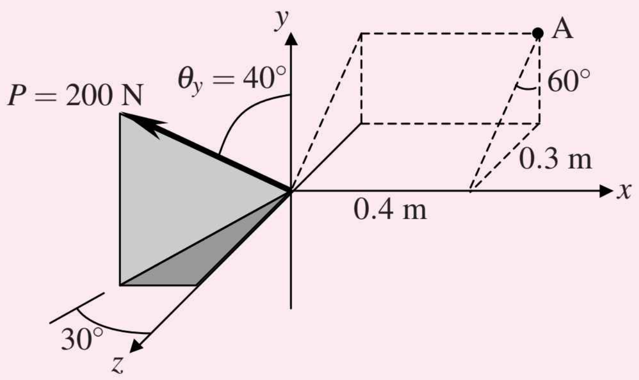


Exercise 7.22

Determine:

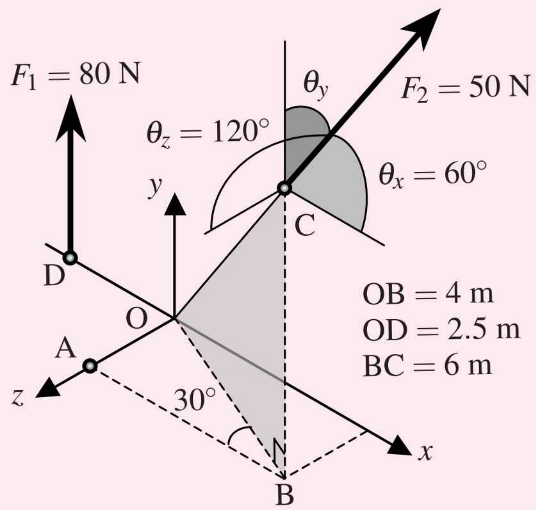
a. The moment of the force $P = 200$ N about point A.

b. The nearest distance between point A and the line of action of force P.



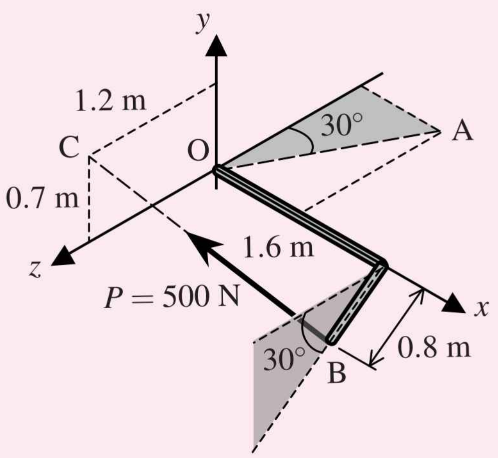
Exercise 7.23

Determine the moment of the two forces F_1 and F_2 shown about point A.



Exercise 7.24

Determine the moment of the 500 N force acting at point B about the OA axis.



8 — Rigid bodies : Couples

8.1 Concept of couples

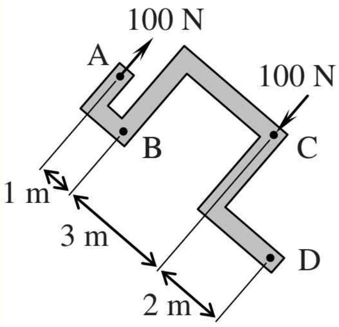
In a simple terms, couple is a pair of forces that have

- the same magnitude
- opposite directions with respect to each other
- parallel line of action

8.1.1 2D couples

The concept of two dimensional couple is better illustrated by the following example.

Example 8.1



From $M = Fd$;

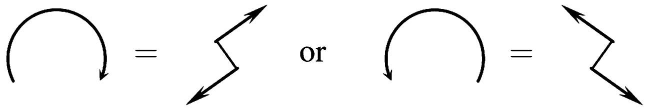
$$(+\circlearrowleft) M_A = (100)(1+3) = 400 \text{ Nm}$$
$$(+\circlearrowleft) M_B = (100)(3) + (100)(1) = 400 \text{ Nm}$$
$$(+\circlearrowleft) M_C = (100)(1+3) = 400 \text{ Nm}$$
$$(+\circlearrowleft) M_D = (100)(6) - (100)(2) = 400 \text{ Nm}$$

Observe that in the example 8.1, moments produced by the couple at different points have the following relationship;

$$M_A = M_B = M_C = M_D.$$

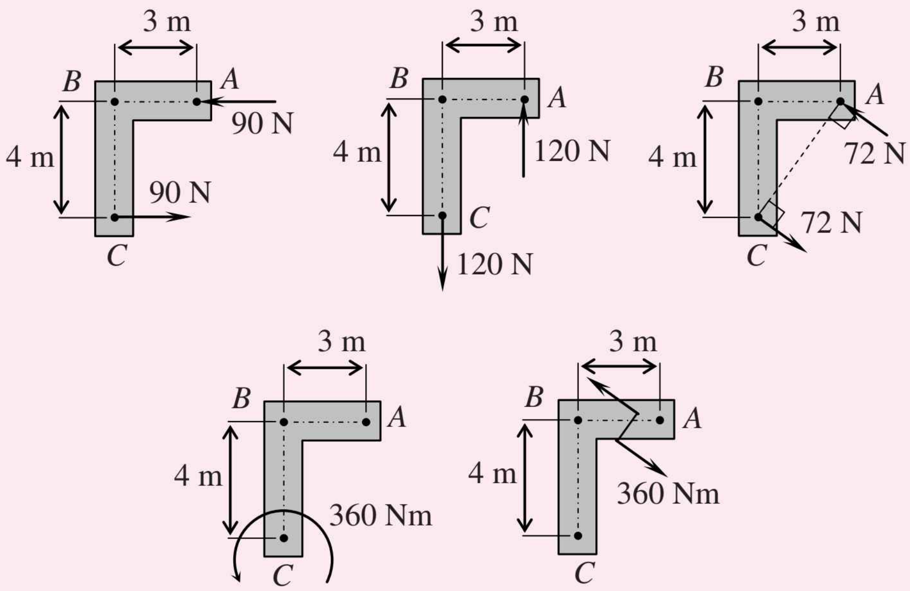
Thus, moment produced by a couple is independent of the point the moment is taken. *The moment from a couple is a free vector, meaning that the moment can be placed anywhere on the rigid body.*

Magnitude of the moment of a couple equals the force magnitude times the perpendicular distance between the line of action of the two forces, $M = F \cdot d$. The moment of a couple can be represented using these symbols;



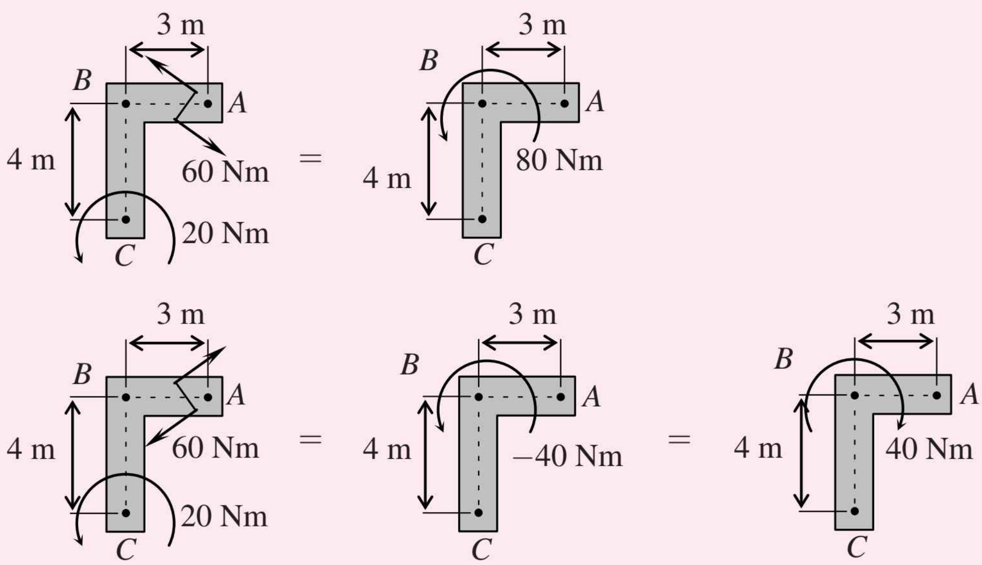
Two couples are the same if the resulting moment are equal (magnitude and direction) AND both couples are on the same or parallel planes.

Concept 8.1 **Equivalent couples (2D)** The following couples are equivalent.



Addition and subtraction of couples in two dimension is made based on the couple rotation directions.

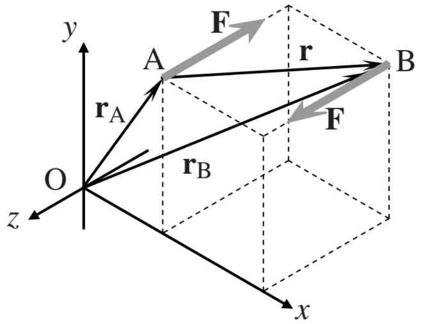
Concept 8.2 **Addition and substraction of couples (2D)**



8.1.2 3D couples

The concept of couples also applies to three dimension as shown in the following example.

■ **Example 8.2**



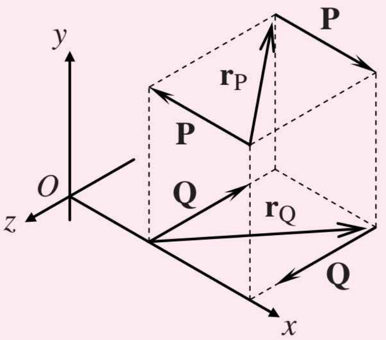
$$\begin{aligned}\mathbf{M}_O &= (\mathbf{r}_A \times -\mathbf{F}) + (\mathbf{r}_B \times \mathbf{F})\\ &= (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}\end{aligned}$$

$$\text{but} \quad \mathbf{r}_B - \mathbf{r}_A = \mathbf{r}$$

$$\therefore \quad \mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

Observe that the moment of the couple is independent of point O.

Concept 8.3 Equivalent couples (3D)



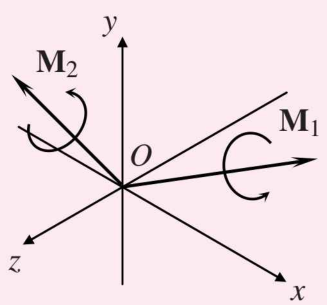
Moment of the couples shown are equivalent if

$$\mathbf{M} = \mathbf{r}_P \times \mathbf{P} = \mathbf{r}_Q \times \mathbf{Q}$$

and make sure that both couples are on the same or parallel planes.

Addition and subtraction of couples (3D) is made by adding/subtracting the components of couples involved.

Concept 8.4 Addition and subtraction of couples (3D)



$$\begin{aligned}\mathbf{M}_1 &= M_{1x}\mathbf{i} + M_{1y}\mathbf{j} + M_{1z}\mathbf{k}\\ \mathbf{M}_2 &= M_{2x}\mathbf{i} + M_{2y}\mathbf{j} + M_{2z}\mathbf{k}\end{aligned}$$

Addition:

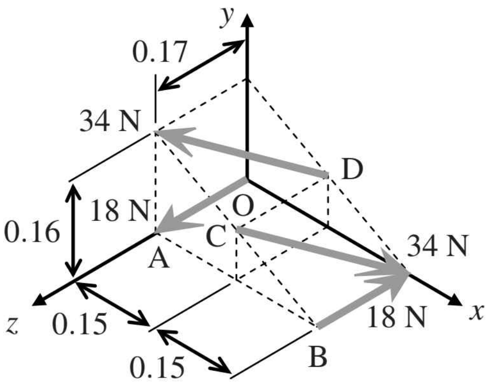
$$\begin{aligned}\mathbf{M} &= \mathbf{M}_1 + \mathbf{M}_2\\ \mathbf{M} &= (M_{1x} + M_{2x})\mathbf{i} + (M_{1y} + M_{2y})\mathbf{j} + (M_{1z} + M_{2z})\mathbf{k}\end{aligned}$$

Subtraction:

$$\begin{aligned}\mathbf{M} &= \mathbf{M}_1 - \mathbf{M}_2\\ \mathbf{M} &= (M_{1x} - M_{2x})\mathbf{i} + (M_{1y} - M_{2y})\mathbf{j} + (M_{1z} - M_{2z})\mathbf{k}\end{aligned}$$

■ **Example 8.3**

Replace the two couples shown in the diagram with one equivalent couple. (all dimensions in m)



Solution:

for the 18 N couple $\Rightarrow \mathbf{M}_1 = \mathbf{r} \times \mathbf{F}$

The 18 N forces are parallel to z-axis. Hence any vectors connecting the force lines can be use as vector $\mathbf{r}$. Following are two simplest choices;

if $\mathbf{F} = +18\text{ N }\mathbf{k}$, then $\mathbf{r} = -0.3\text{ m }\mathbf{i}$
if $\mathbf{F} = -18\text{ N }\mathbf{k}$, then $\mathbf{r} = +0.3\text{ m }\mathbf{i}$

both options will produce the same result. Thus;

$$\mathbf{M}_1 = (-0.3\text{ m }\mathbf{i}) \times (+18\text{ N }\mathbf{k}) = (+0.3\text{ m }\mathbf{i}) \times (-18\text{ N }\mathbf{k}) = 5.4\text{ Nm }\mathbf{j}$$

for the 34 N couple $\Rightarrow \mathbf{M}_2 = \mathbf{r} \times \mathbf{F}$

determine the components of force $\mathbf{F}$;

$$\mathbf{F} = F\boldsymbol{\lambda} = F\left(\frac{dx}{d}\mathbf{i} + \frac{dy}{d}\mathbf{j} + \frac{dz}{d}\mathbf{k}\right)$$

taking F at C; thus

$$\begin{aligned} d &= \sqrt{0.15^2 + 0.08^2 + 0.17^2} = 0.2404 \\ \mathbf{F} &= (34\text{ N})\left(\frac{0.15}{0.2404}\mathbf{i} + \frac{-0.08}{0.2404}\mathbf{j} + \frac{-0.17}{0.2404}\mathbf{k}\right) \\ &= 21.2\text{ N }\mathbf{i} - 11.3\text{ N }\mathbf{j} - 24\text{ N }\mathbf{k} \end{aligned}$$

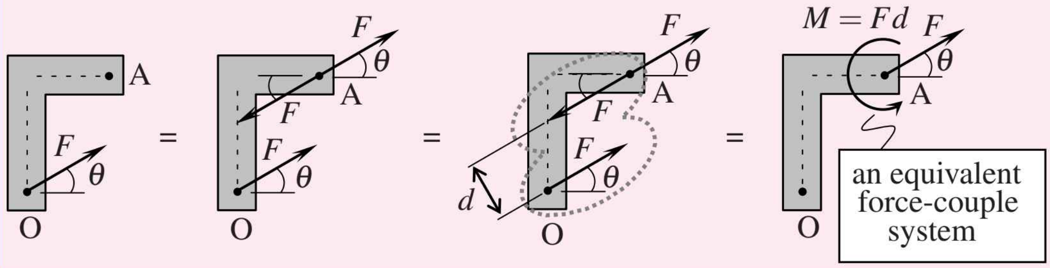
position vector $\mathbf{r}$ from D to C is $\mathbf{r} = 0.17\text{ m }\mathbf{k}$. Therefore;

$$\begin{aligned} \mathbf{M}_2 &= (0.17\text{ m }\mathbf{k}) \times (21.2\text{ N }\mathbf{i} - 11.3\text{ N }\mathbf{j} - 24\text{ N }\mathbf{k}) \\ &= (0.17\text{ m })(21.2\text{ N})(\mathbf{j}) + (0.17\text{ m })(-11.3\text{ N})(-\mathbf{i}) \\ &= 1.921\text{ Nm }\mathbf{i} + 3.604\text{ Nm }\mathbf{j} \\ \therefore \mathbf{M} &= \mathbf{M}_1 + \mathbf{M}_2 = 5.4\text{ Nm }\mathbf{j} + 1.921\text{ Nm }\mathbf{i} + 3.604\text{ Nm }\mathbf{j} \\ &= 1.921\text{ Nm }\mathbf{i} + 9.004\text{ Nm }\mathbf{j} \end{aligned}$$

8.1.3 2D force-couples system

This concept is used to move force from one location to the other location. For example to move a force from O to A as shown in the following;

Concept 8.5 **Force-couple system**



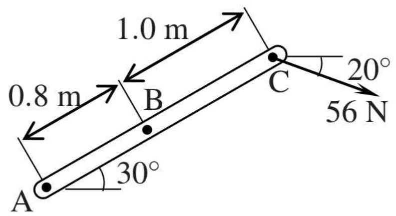
Or simpler, to move a force to a different location (from O to A for example)

- Find the moment of the force about point A.
- Detach the force and place it at point A.

Example 8.4

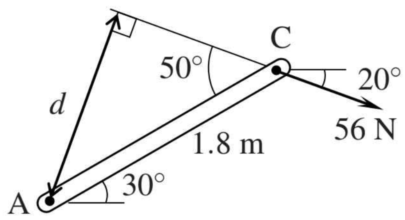
Replace the force at C with an equivalent force–couple system at

(a) point A,
(b) point B.

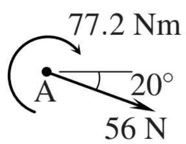


Solution (a):

$$(+\circlearrowright)M_A = Fd$$
$$= (56\ \text{N})(1.8 \sin 50^\circ\ \text{m})$$
$$= 77.2\ \text{Nm}$$



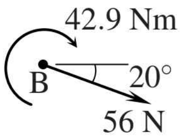
∴ the force-couple system at A:

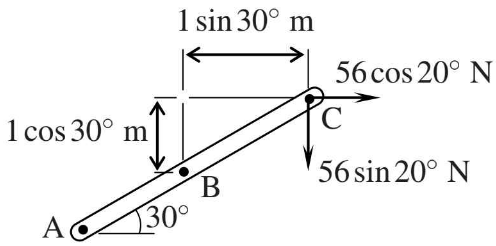


Solution (b):

$$\begin{aligned}(+\circlearrowright)M_B &= Fd\\ &= (56\cos 20^\circ\ \text{N})(1.0\sin 30^\circ\ \text{m})\\ &\quad + (56\sin 20^\circ\ \text{N})(1.0\cos 30^\circ\ \text{m})\end{aligned}$$

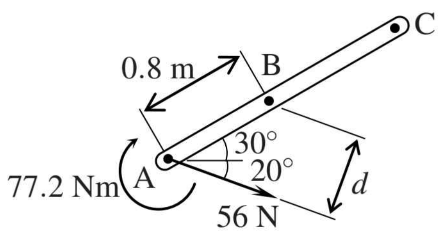
∴ the force-couple system at B:



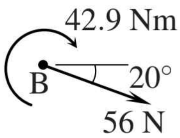


The force–couple system at B can also be found starting from the force–couple system at A because the systems are equivalent. Therefore;

$$\begin{aligned}(+\circlearrowright)M_B &= Fd\\ &= 77.2 - (56\ \text{N})(0.8\sin 50^\circ\ \text{m})\\ &= 42.9\ \text{Nm}\end{aligned}$$

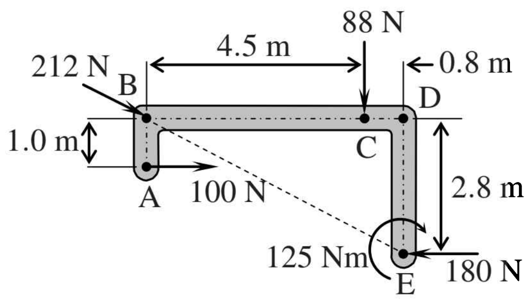


∴ the force-couple system at B:



■ **Example 8.5**

Replace the system of forces and a couple acting on the rigid body with an equivalent force–couple system at point D.



Solution:

Determine the angle of the 212 N force;

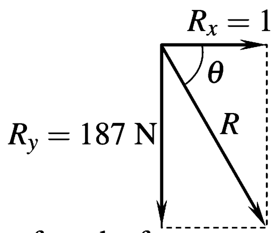
$$\angle\text{CBE} = \tan^{-1}\frac{2.8}{5.3} = 27.85^\circ$$

Taking moment about point D;

$$
\begin{aligned}
(+\circlearrowright)M_D &= +(180\text{ N})(2.8\text{ m}) - (88\text{ N})(0.8\text{ m}) - (212\sin 27.85^\circ\text{ N})(5.3\text{ m}) \\
&\quad - (100\text{ N})(1.0\text{ m}) + (125\text{ Nm}) \\
&= -66.3\text{Nm} \\
\therefore M_D &= 66.3\text{Nm}(\circlearrowleft)
\end{aligned}
$$

Determine resultant of the forces;

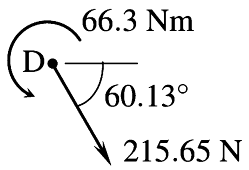
$$
\begin{aligned}
(\rightarrow +)\sum F_x &= R_x \\
R_x &= 212\cos 27.85^\circ\text{ N} + 100\text{ N} - 180\text{ N} = 107.4\text{ N} \\
(\uparrow +)\sum F_y &= R_y \\
R_y &= -212\sin 27.85^\circ\text{ N} - 88\text{ N} = -187\text{ N} \\
\therefore R_y &= 187\text{ N}(\downarrow)
\end{aligned}
$$



$$
R = \sqrt{107.4^2 + 187^2} = 215.65\text{ N}
$$

$$
\theta = \tan^{-1}\frac{187}{107.4} = 60.13^\circ
$$

Therefore the force–couple system at point D is;



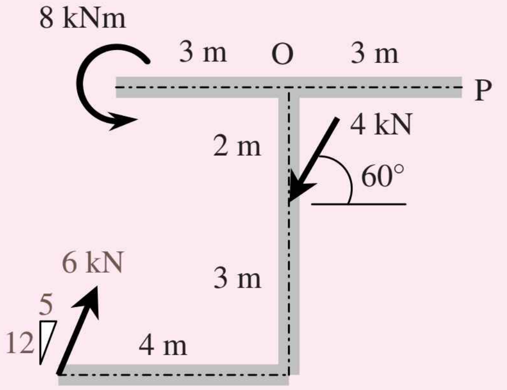
From the previous example, it can be concluded that the procedure to reduce a system of forces to a single force–couple system (i.e. at point A) are;

i. Find the moment of all forces about point A.
ii. Determine the resultant of all forces and place it at point A.

8.2 Example questions

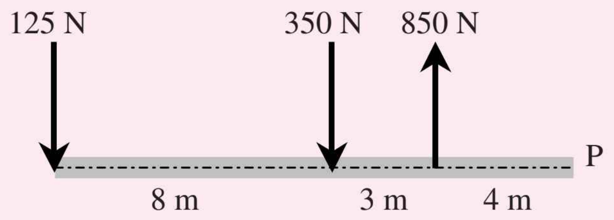
Exercise 8.1

Replace the forces and couple system in the figure with an equivalent force–couple system at P.
Answer: $R_P = 2.097$ kN ($\theta = 81.6°$ ∡) and $M_P = 12.84$ kNm (↻)



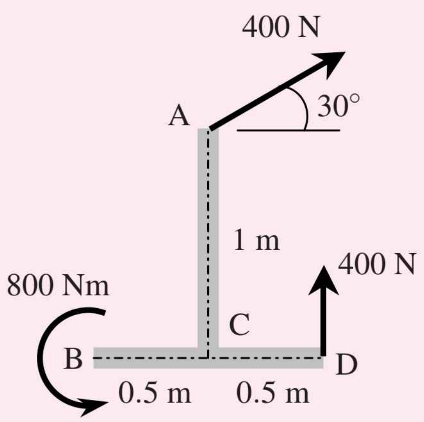
Exercise 8.2

Replace the system of forces in the figure with an equivalent force–couple system at P.
Answer: $R_P = 375$ N (↑) and $M_P = 925$ Nm (↺)



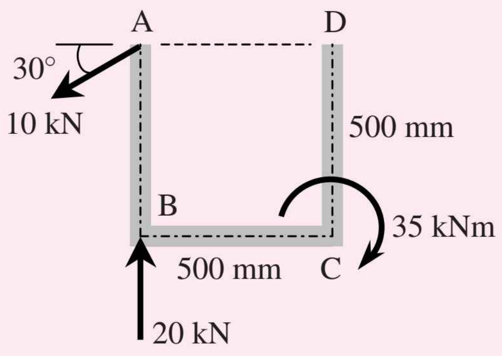
Exercise 8.3

Replace the forces and couple system in the figure with an equivalent force–couple system at C.



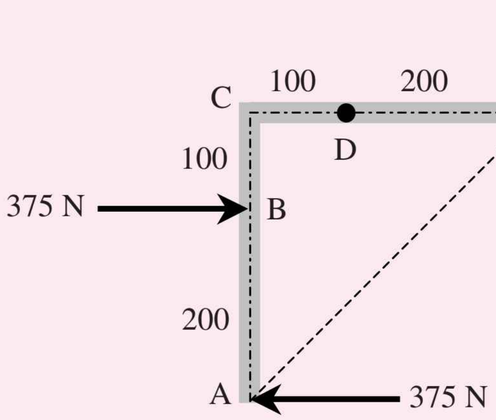
Exercise 8.4

Replace the system of forces and couples in the figure with an equivalent force–couple system at A.



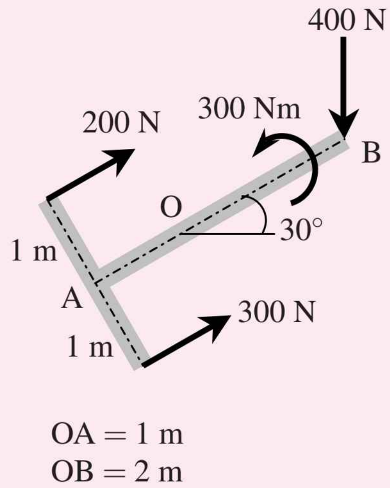
Exercise 8.5

Replace the system of forces in the figure with an equivalent force–couple system at D. All dimensions in mm.



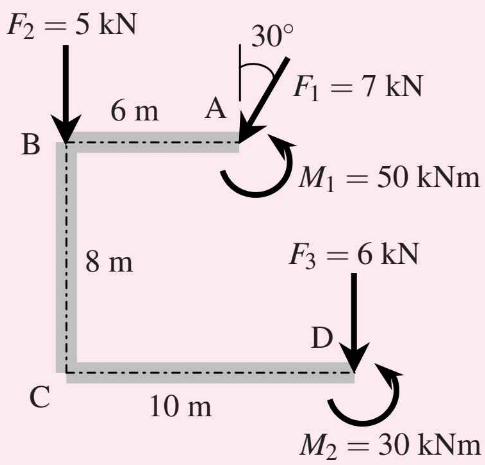
Exercise 8.6

Replace the system of forces in the figure with an equivalent force–couple system at O.



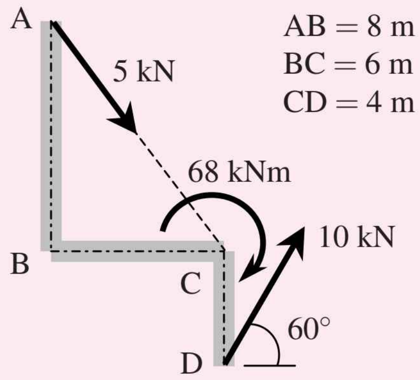
Exercise 8.7

Replace the forces and couples system in the figure with an equivalent force–couple system at C.



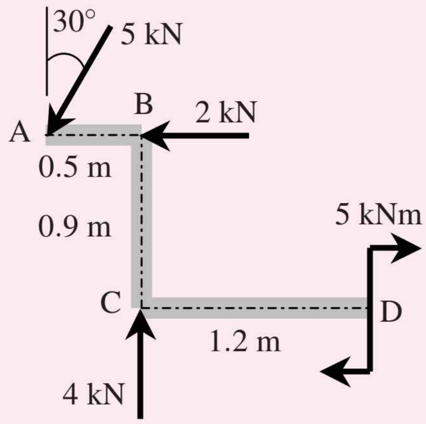
Exercise 8.8

Replace the forces and couple system in the figure with an equivalent force–couple system at A.



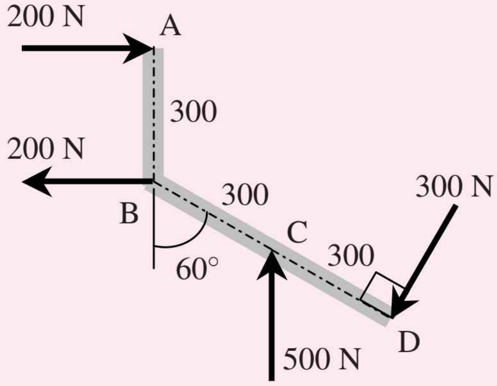
Exercise 8.9

Replace the forces and couple system in the figure with an equivalent force–couple system at B.
Answer: $R_B = 4.512$ kN $(\theta = 4.2^\circ)$ and $M_B = 2.83$ kNm



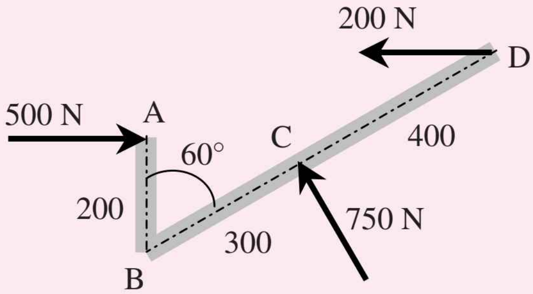
Exercise 8.10

Replace the system of forces in the figure with an equivalent force–couple system at B. All dimensions in mm.

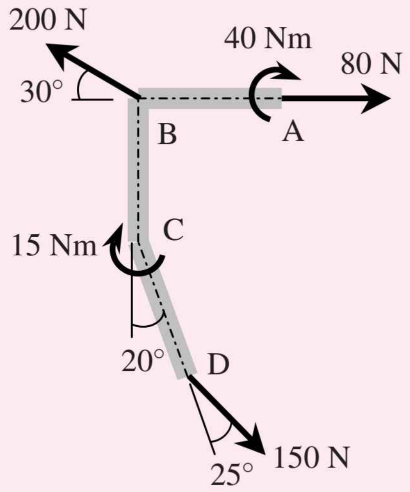


Exercise 8.11

Replace the system of forces in the figure with an equivalent force–couple system at B. All dimensions in mm.



Exercise 8.12

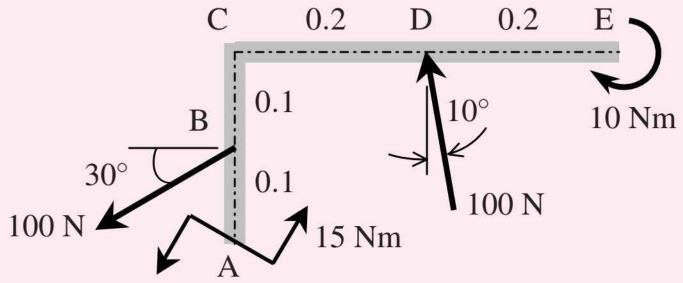


Replace the system of forces and couples in the figure with an equivalent force–couple system at A. Given AB = BC = CD = 0.2 m.

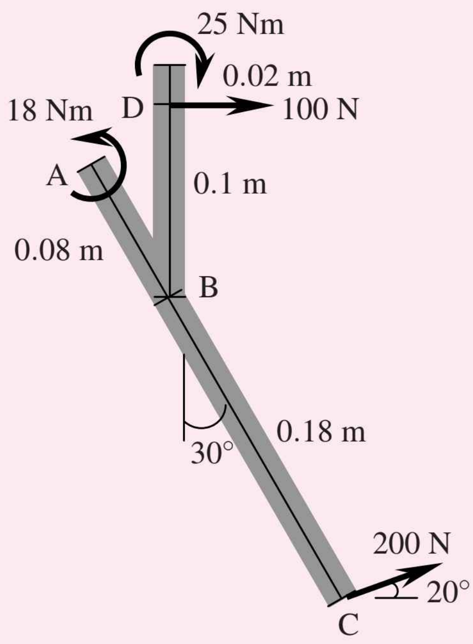
Exercise 8.13

Replace the system of forces and couples in the figure with an equivalent force–couple system at D. All dimensions in m.

Answer: $R_D = 114.8$ N ($\theta = 25°$) and $M_D = 6.34$ Nm (↺)



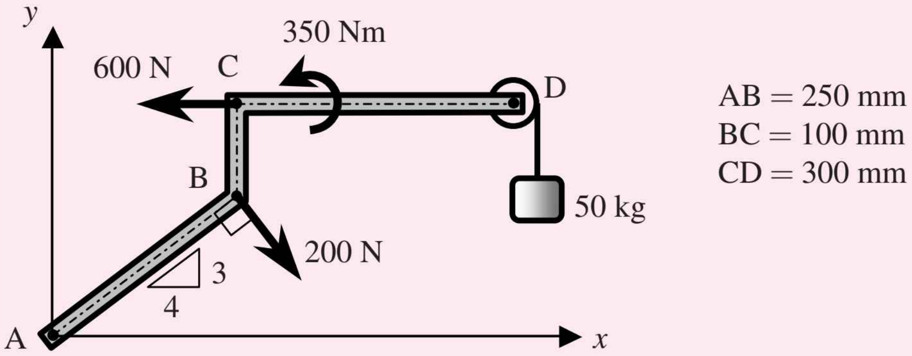
Exercise 8.14



Replace the system shown in the figure with an equivalent force-couple system at B.

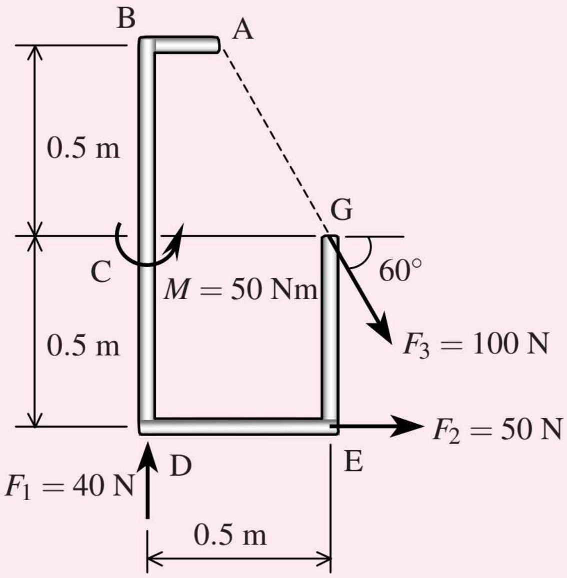
Exercise 8.15

Replace the system of forces and couple in the figure with an equivalent force–couple system at A.



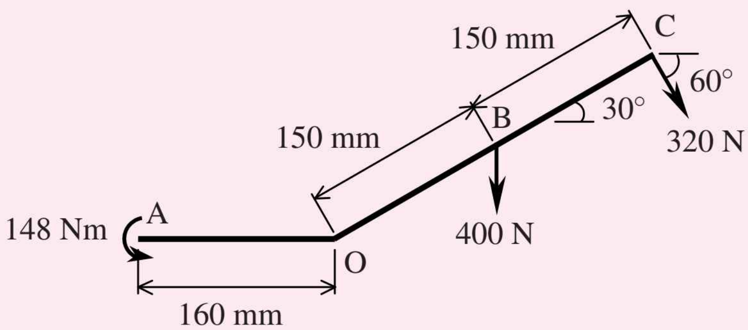
Exercise 8.16

Replace the system of forces and couples in the figure with an equivalent force–couple system at B.



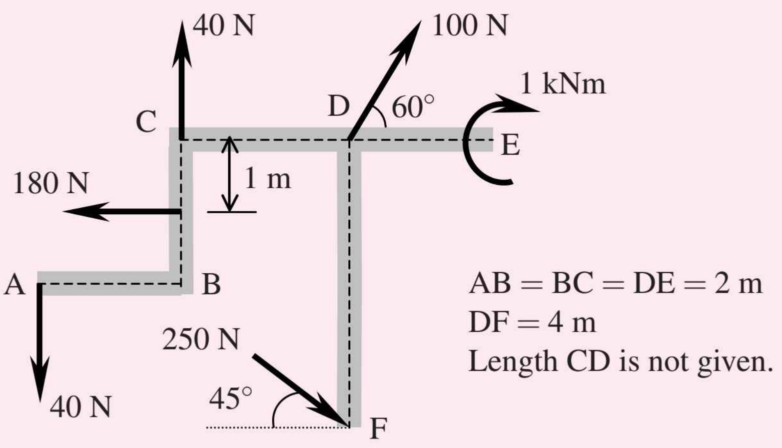
Exercise 8.17

Rigid body AOBC is acted upon by the forces and couple as shown. Replace the system with an equivalent force–couple system at point O.



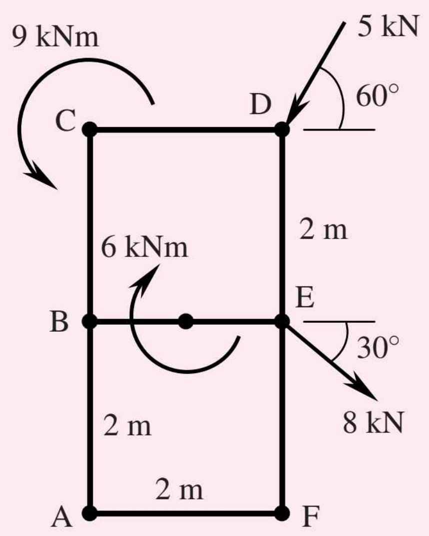
Exercise 8.18

The rigid body ABCDEF is acted upon by forces and couples as shown in the diagram. Replace the system with an equivalent force-couple system at E.



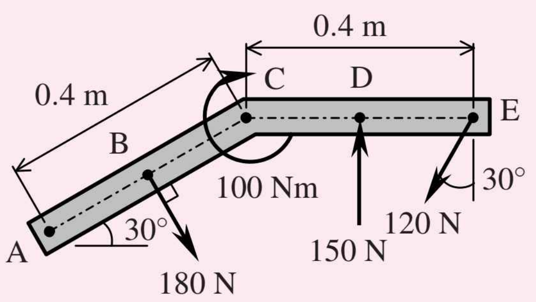
Exercise 8.19

A rigid body supports the loading shown. Determine an equivalent force–couple system at A.



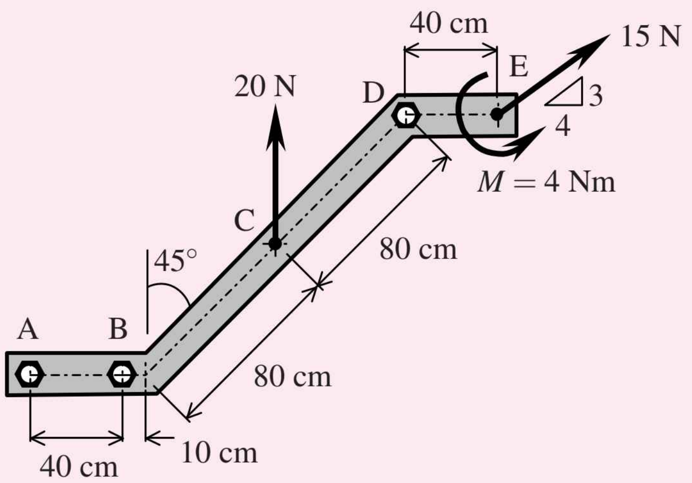
Exercise 8.20

A rigid body supports the loading shown. Replace the given system with an equivalent force–couple system at A. Points B and D are mid points of AC and CE respectively.



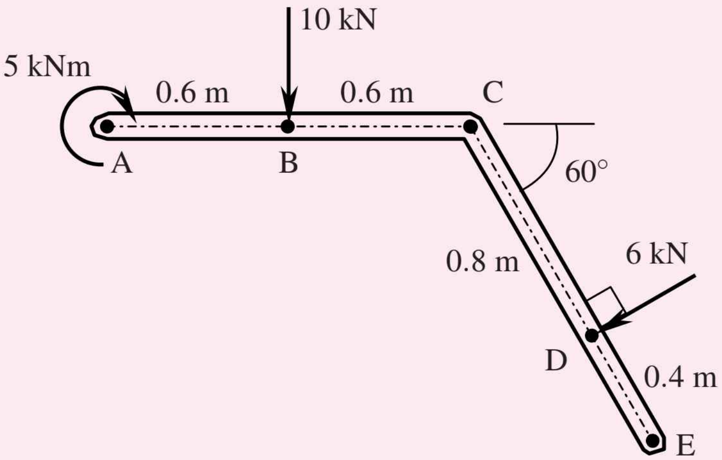
Exercise 8.21

The figure shows a rigid body ABCDE acted upon by two forces and a couple. Determine an equivalent force-couple system at A.



Exercise 8.22

A couple of 5 kNm and two forces of magnitude 10 kN and 6 kN acted on the rigid body shown. Determine an equivalent force couple system at C.



9 — Rigid bodies : Equilibrium 2D

9.1 Equilibrium condition in 2D

The differences between equilibrium conditions of particle and rigid body in two dimension (i.e. x-y plane) are shown in the following table;

	Equilibrium conditions
Particles	$\sum F_x = 0$ and $\sum F_y = 0$
Rigid bodies	$\sum F_x = 0$ and $\sum F_y = 0$ and $\sum M_A = 0$ (where A is any point)

For solving the equilibrium conditions, free body diagram (FBD) of the rigid body is needed to identify the relevant forces and moments.

9.2 FBD for 2D rigid bodies

Procedure for drawing FBD is;

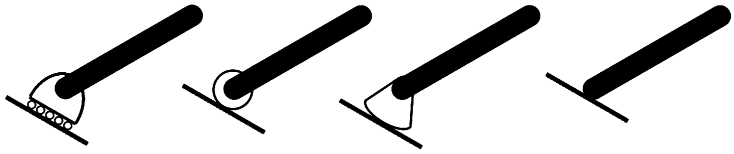
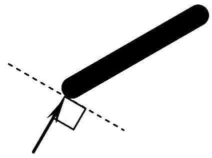
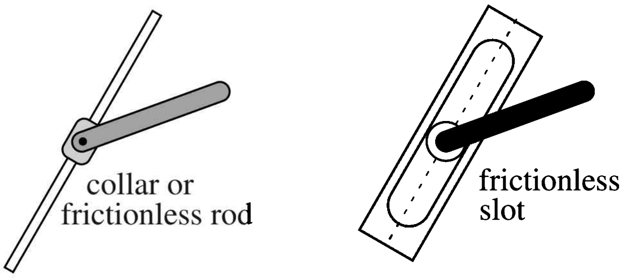
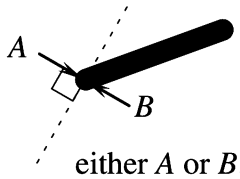
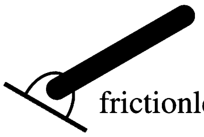
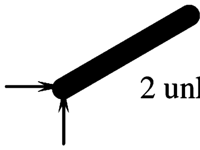
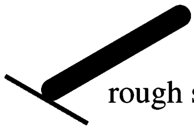
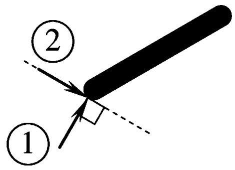
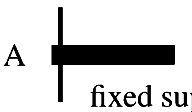
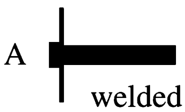
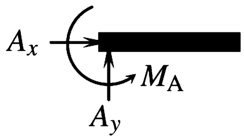
- Draw the boundary of the chosen section and detach/separate it from all other bodies,
- For body with a mass, put the weight $= mg$ acting at the centre of gravity (G) of the body in the vertically downwards direction,
- Input all <u>external</u> forces acting on the body, For systems involving cables, the force of the cable must act outwards of the body, i.e. the cable must be in <u>tension</u>,
- Put the reaction/s where the body touches or connected to a different rigid body,
- The FBD should also include dimensions for the process of taking moment.

When supports or connections on the rigid body are removed, they are replaced with reaction forces and/or reaction moments. These information are summarized in Tables 9.1 and 9.2.

Table 9.1: Reaction forces/moments for respective supports/connections in 2D

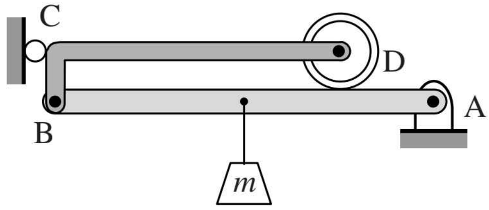
Supports / Connections	**Reaction forces / moments**	**Notes on reaction forces / moments**
• Rollers • Rockers • Frictionless surface	A force with a known line of action	Perpendicular to the surface and must point towards the free body
• Collars on frictionless rod • Frictionless slot • Double track rollers (reversible single track rollers and rockers)	A force with a known line of action	Perpendicular to the rod/slot and can be directed either way (not both)
• Frictionless pin	Two force components, involving 2 unknowns	Usually represented by their x and y components that are normally assumed
• Rough surface	Two force components, involving 2 unknowns	One perpendicular to the surface and must point towards the free body and the other 90° to it (tangent to the surface) that can be directed either way (not both)
• Fixed support • Welded joint	Two force components and one couple, involving 3 unknowns	The force components are usually represented by their x and y components that are normally assumed. The sense of the couple can be arbitrarily assumed, the sign of the answer will indicate the correct direction

Table 9.2: Summary of reaction forces/moments in 2D

Supports / Connections	**Reactions**
 rollers rollers rockers frictionless surface	 90° to the surface and towards the body
	 either A or B, NOT both
 frictionless pin	 2 unknowns
 rough surface	 2 unknowns : 1. 90° to the surface and towards the body 2. Either upwards or downwards the incline
 fixed support  welded support	

■ Example 9.1

Draw the Free Body Diagram for the mechanism shown. Mass of bodies is negligible unless stated by m. All contacting surfaces are smooth unless otherwise stated.

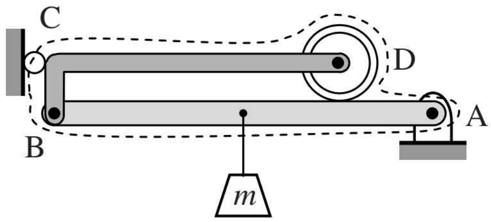


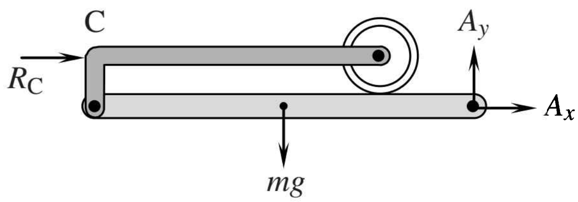
Solution:

Overall FBD

The procedure is as follows;

- Draw the boundary of the required body
- Put the weight $W = mg$ when m is given/known
- There are no external forces in this example
- Reactions:
  - Remove pin joint at A $\rightarrow$ 2 components of reaction, A_x & A_y
  - Remove roller at C $\rightarrow$ 1 force directed to the body, R_C
  - No reaction at B because the pin is not removed
  - No reaction at D because the roller is not separated from body (considered as one unit)
  - No reaction at pin at centre of roller because the pin is not removed

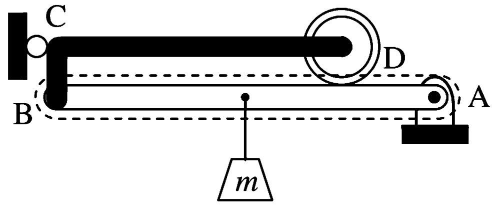


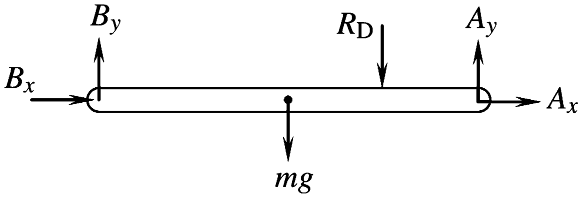


FBD of member AB

The procedure is as follows;

- Draw the boundary of the required body
- Put the weight $W = mg$ when m is given/known
- There are no external forces in this example
- Reactions:
  - Remove pin joint at A $\rightarrow$ 2 components of reaction, A_x & A_y
  - Remove pin joint at B $\rightarrow$ 2 components of reaction, B_x & B_y
  - Remove roller at D $\rightarrow$ 1 force directed to the body, R_D

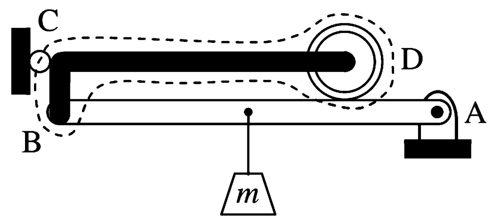


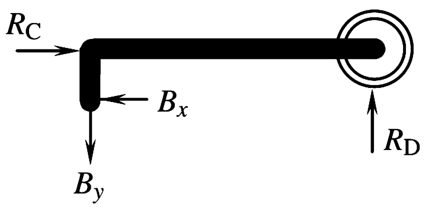


FBD of member BCD

The procedure is as follows;

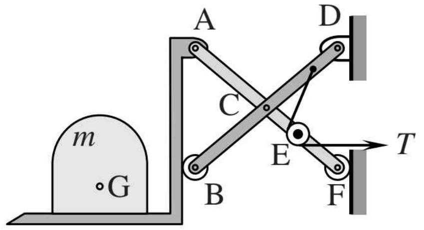
- Draw the boundary of the required body
- No m is given/known
- There are no external forces in this example
- Reactions:
  - Remove pin joint at B $\rightarrow$ 2 components of reaction, B_x & B_y
  - Remove roller at D $\rightarrow$ 1 force directed to the body, R_D
  - Remove roller at C $\rightarrow$ 1 force directed to the body, R_C
  - No reaction at pin at centre of roller because the pin is not removed at D





■ Example 9.2

Draw the free body diagram for the mechanism shown. Mass of bodies are negligible unless stated by m. All contacting surfaces are smooth unless otherwise stated.

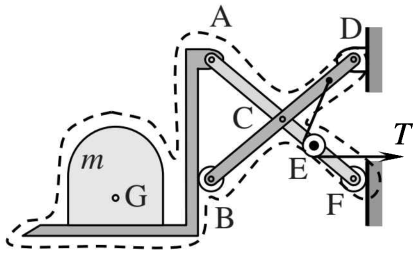


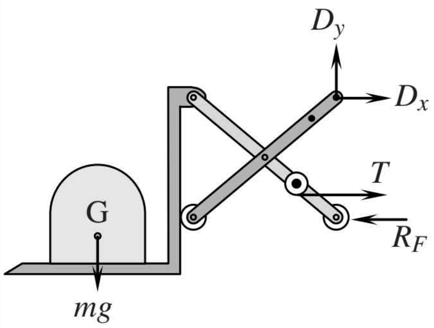
Solution:

Overall FBD

The procedure is as follows;

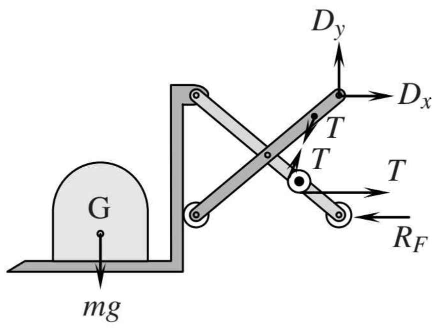
- Draw the boundary of the required body
- Put the weight $W = mg$ when m is given/known.
- The external force, the cable, must direct outwards of the body. Since part of the cable exit and enter the boundary, the two forces cancel each other, as shown in the solution below.
- Reactions:
  - Remove pin joint at D
    $\rightarrow$ 2 components of reaction, D_x & D_y
  - Remove roller at F
    $\rightarrow$ 1 force directed to the body, R_F
  - No reaction at other pins and roller because they are not removed





Most popular solution

OR

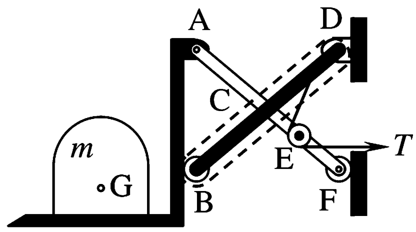


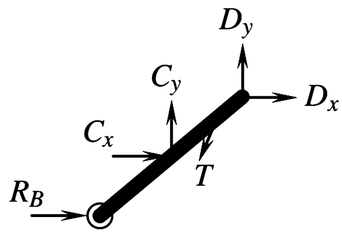
Acceptable solution, but unnecessary

Free Body Diagram of member BCD

The procedure is as follows;

- Draw the boundary of the required body
- No *m* is given/known.
- The external force, the cable, must direct outwards of the body.
- Reactions:
  - Remove pin joint at D
    → 2 components of reaction, D_x & D_y
  - Remove pin joint at C
    → 2 components of reaction, C_x & C_y
  - Remove roller at B
    → 1 force directed to the body, R_B

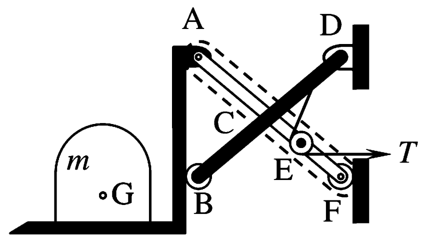


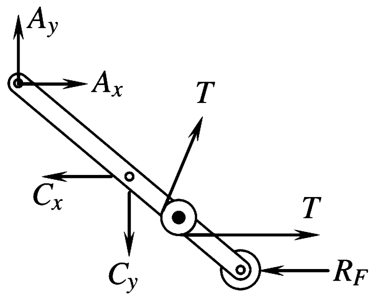


Free Body Diagram of member ACEF

The procedure is as follows;

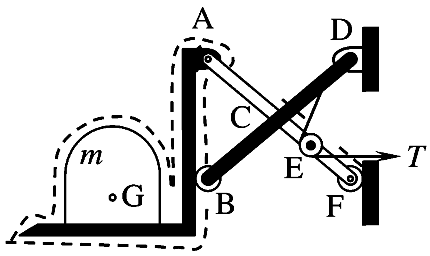
- Draw the boundary of the required body
- No *m* is given/known.
- The external force, the cable, must direct outwards of the body.
- Reactions:
  - Remove pin joint at A
    → 2 components of reaction, A_x & A_y
  - Remove pin joint at C
    → 2 components of reaction, C_x & C_y
  - Remove roller at F
    → 1 force directed to the body, R_F



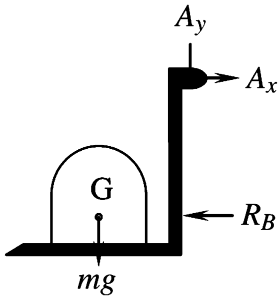


Free Body Diagram of member ABG

The procedure is as follows;



- Draw the boundary of the required body
- Put the weight $W = mg$ when m is given/known.
- There are no external forces in.
- Reactions:
  - Remove pin joint at A
    $\rightarrow$ 2 components of reaction, A_x & A_y
  - Remove roller at B
    $\rightarrow$ 1 force directed to the body, R_B



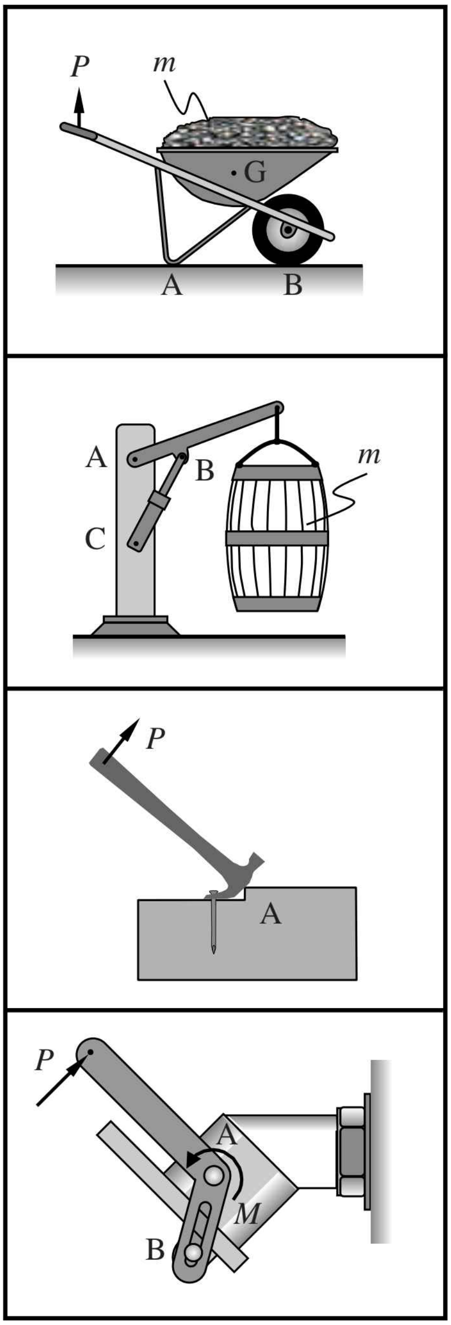
> The sense (direction) of force components at the pins are **arbitrarily assumed**.

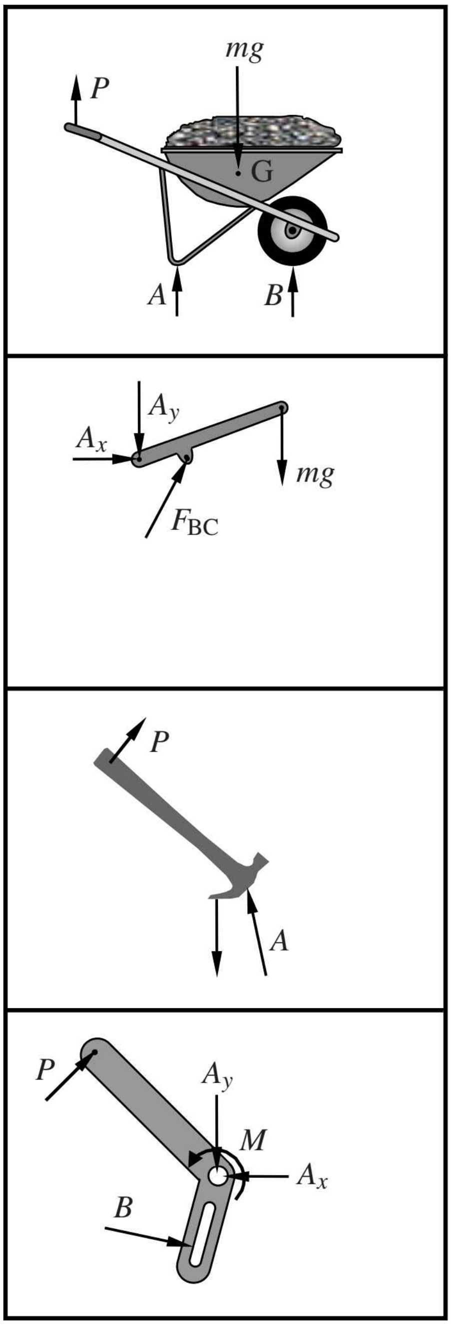
■ Example 9.3

Taking into consideration Newton's Third Law and two force members where applicable, complete the Free Body Diagrams on the right. Mass of bodies is stated as m and all contacting surfaces are smooth unless stated otherwise.

Solution:

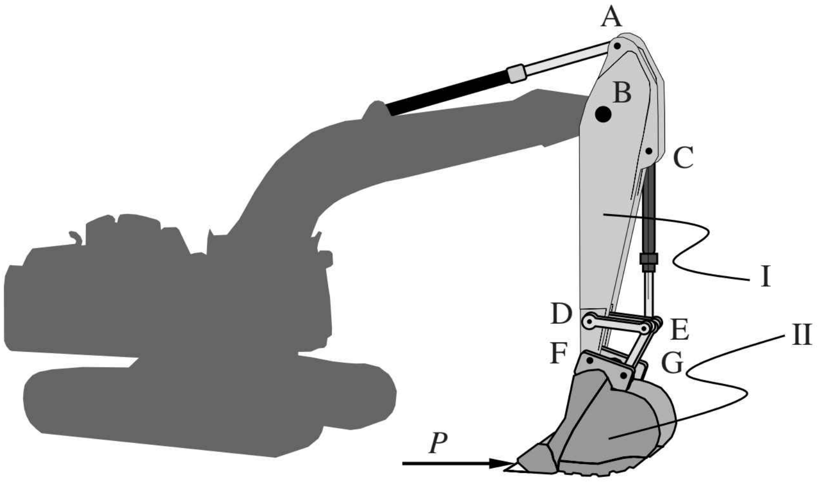
Free body diagrams





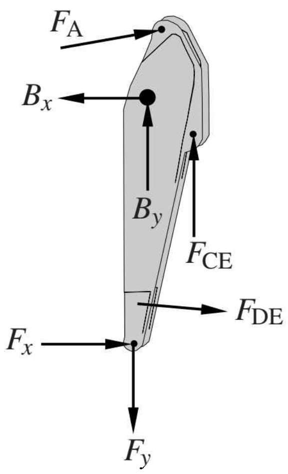
■ Example 9.4

Taking into consideration of Newton's Third Law and two force members where applicable, draw and complete the Free Body Diagram (in 2 dimension) for members I and II of the excavator shown. Mass of members is negligible compared to load P. All contacting surfaces are smooth unless stated otherwise.

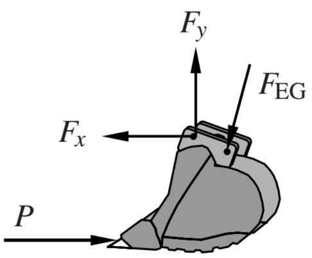


Solution:

Member I



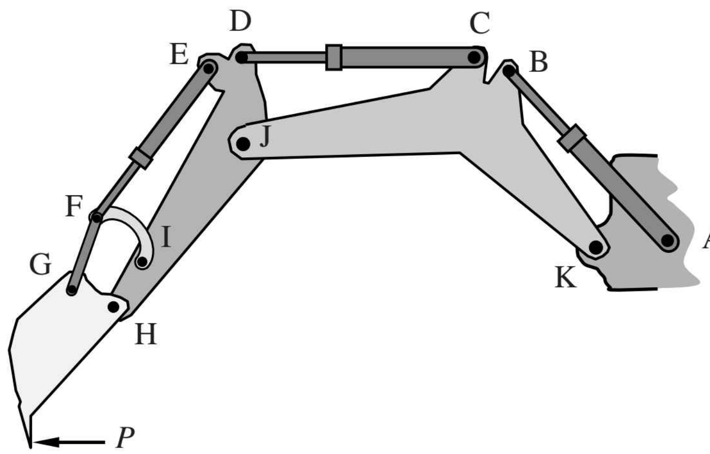
Member II



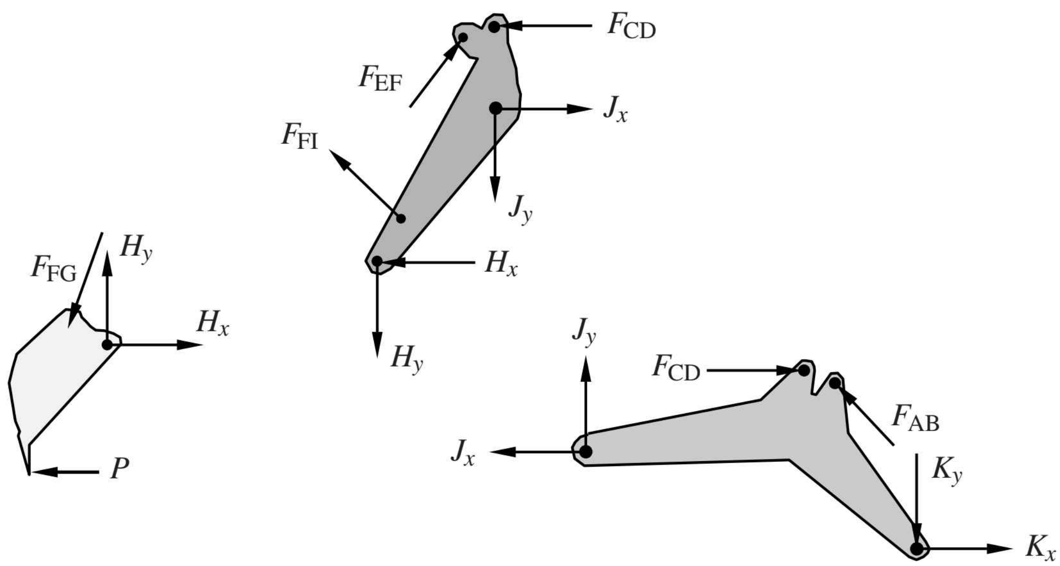
■ **Example 9.5**

Taking into consideration Newton's Third Law and two force members where applicable, draw and complete the Free Body Diagrams of the main bodies of the mechanism. Mass of all members is considered negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

Main Bodies:
BCJK, DEHIJ and GH

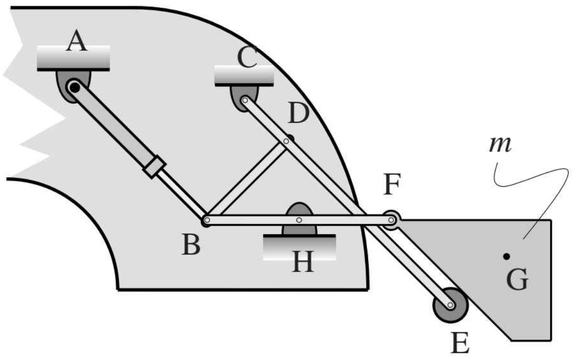


Solution:

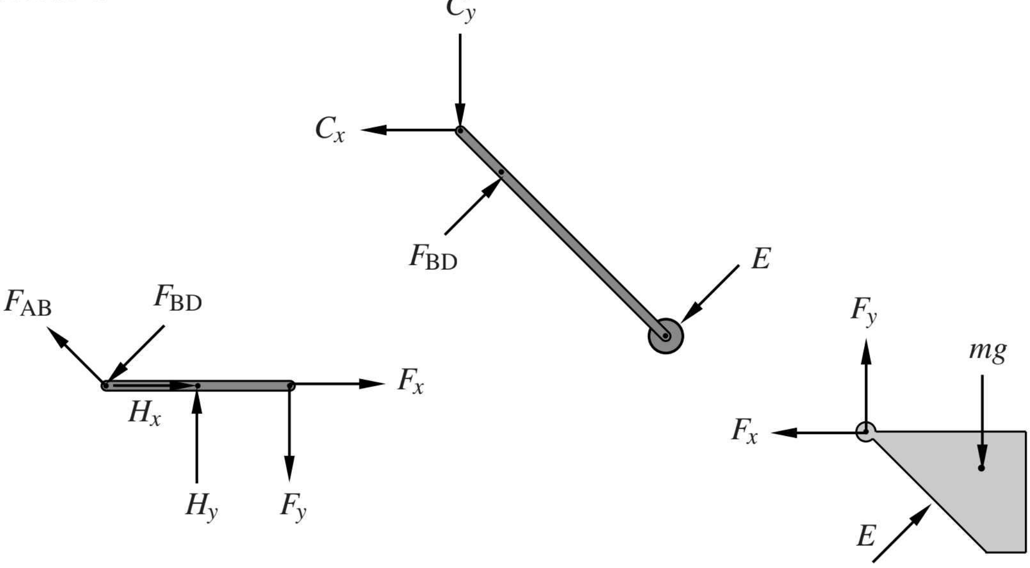


■ Example 9.6

Taking into consideration Newton's Third Law and two force members where applicable, draw and complete the Free Body Diagrams of the main bodies of the mechanism. Mass of all members is considered negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.



Solution:



9.3 Analysis approach

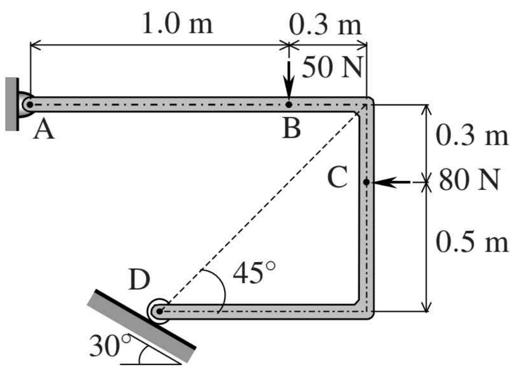
Procedures for solving the 2D rigid body equilibrium problems are as follows;

- First, sketch the Free Body Diagram.
- Solve the problem based on the equilibrium conditions ($\sum F = \sum M = 0$) either using equations;
  - (i) $\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_A = 0$ (arbitrary point A)
  - (ii) $\sum F_x = 0, \quad \sum M_A = 0, \quad \sum M_B = 0$ (arbitrary points A and B)
  - (iii) $\sum F_y = 0, \quad \sum M_A = 0, \quad \sum M_B = 0$ (arbitrary points A and B)
  - (iv) $\sum M_A = 0, \quad \sum M_B = 0, \quad \sum M_C = 0$ (arbitrary points A, B and C)

Moments can be taken at any points, even from points outside the body. Since there are only 3 equations, there must be **only 3 unknowns** in the problem!

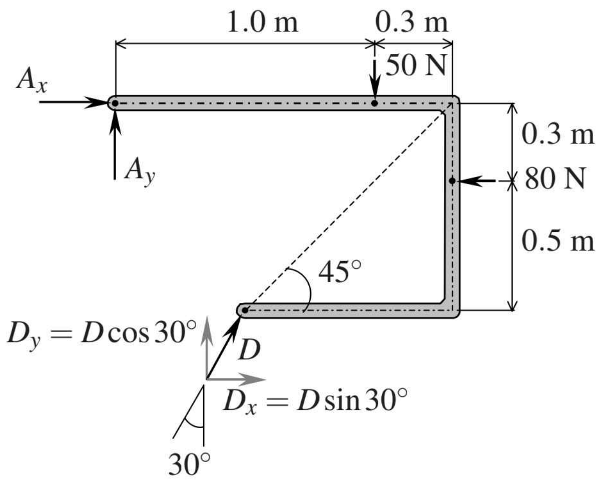
Example 9.7

Determine the reactions at A and D of the rigid body shown in the diagram.



Solution:

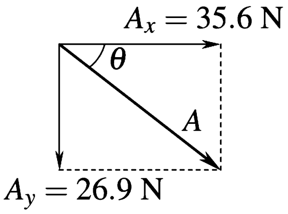
Draw FBD of the rigid body;



The equations from the equilibrium condition are used to solve the problem. The common practice (not necessarily) is to use the moment equation first, taking moment about the point of intersection of two lines of action of 2 unknowns, directly solving for one unknown. By taking moment about point A and taking summation of forces along x and y axes yield;

$$
\begin{aligned}
(+\circlearrowright) M_\text{A} &= 0 \\
&= +(50\text{ N})(1\text{ m}) + (80\text{ N})(0.3\text{ m}) - (D\cos 30^\circ\text{ N})(1.3-0.8) \\
&\quad -(D\sin 30^\circ\text{ N})(0.3+0.5) = 0 \\
\therefore D &= 88.8\text{ N} \quad \measuredangle 30^\circ \\
(+\rightarrow)\sum F_x &= A_x - 80 + D\sin 30^\circ = 0 \\
&= A_x - 80 + (88.8)\sin 30^\circ = 0 \\
\therefore A_x &= 35.6\text{ N} \quad (\rightarrow) \\
(+\uparrow)\sum F_y &= A_y - 50 + D\cos 30^\circ = 0 \\
&= A_y - 50 + (88.8)\cos 30^\circ = 0 \\
\therefore A_y &= -26.9\text{ N} \quad (\uparrow) \\
&= 26.9\text{ N} \quad (\downarrow)
\end{aligned}
$$

Solve for reaction at A;

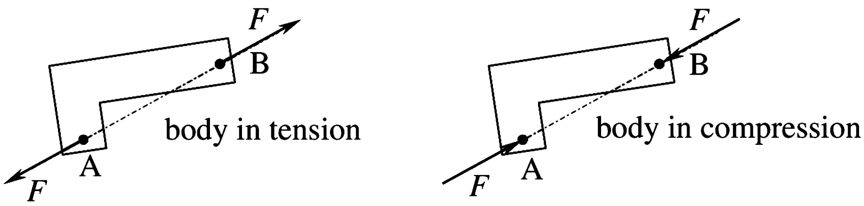


$$
\begin{aligned}
A &= \sqrt{A_x^2 + A_y^2} = \sqrt{35.6^2 + 26.9^2} = 44.6\text{ N} \\
\theta &= \tan^{-1}\frac{A_y}{A_x} = \tan^{-1}\frac{26.9}{35.6} = 37.1^\circ \\
\therefore A &= 44.6\text{ N} \quad \measuredangle 37.1^\circ
\end{aligned}
$$

9.4 Two force member

Consider the rigid body shown with forces acting at point A and B (2 points only). For equilibrium, $\sum M = 0$. At point A, for $\sum M_A = 0$, the force at B must act along the line AB. On the other hand, at point B, for $\sum M_B = 0$, the force at A must act along the line AB as well. Therefore, the two forces must act along the line AB as shown.

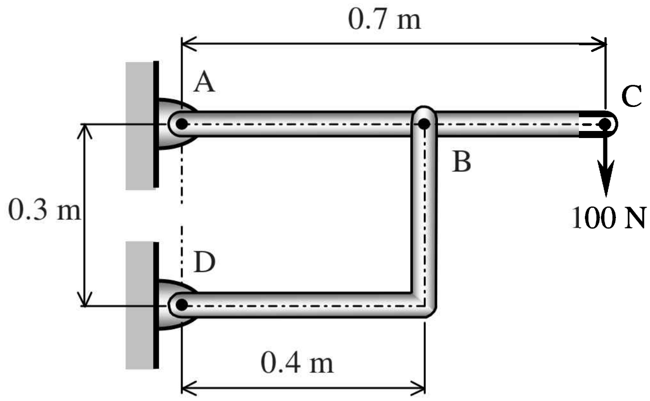
The are two possibilities for the forces, either causing the body to be in tension or compression. For equilibrium, $\sum F = 0$. Thus, both forces at point A and B must have the same magnitude but in opposite sense.



A two force member is a rigid body of **negligible mass** with **only two forces** acting on it.

■ **Example 9.8**

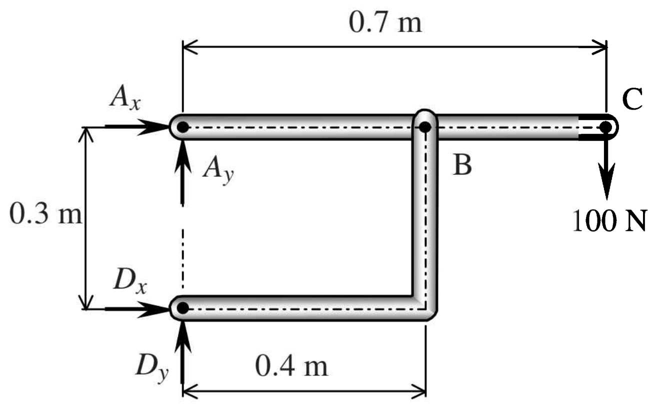
Determine the reactions at A and D of the rigid body shown in the diagram.

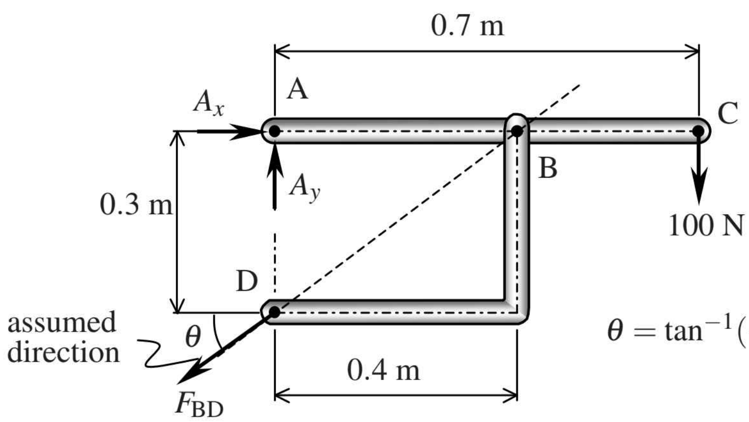


Solution:

Draw the FBD;

Observe that there are 4 unknowns. Note that BD is a two–force member, with a known line of action. Thus by knowing this, the new FBD is as follows;





$$\theta = \tan^{-1}\left(\frac{0.3}{0.4}\right) = 36.87^{\circ}$$

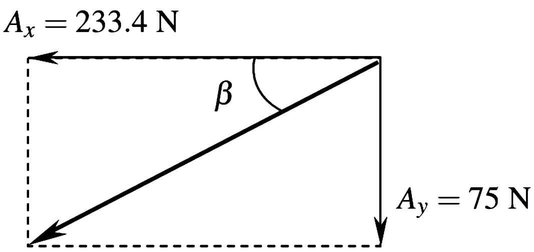
Solve using the equilibrium conditions. Moments is taken at point A (intersection point of 2 unknowns).

$$
\begin{aligned}
(+\circlearrowright) M_{\mathrm{A}} &= 0 \\
&= +100(0.7) + F_{\mathrm{BD}} \cos\theta (0.3) = 0 \\
\therefore F_{\mathrm{BD}} &= -291.7 \text{ N} \\
&= 291.7 \text{ N} \quad \measuredangle 36.87^\circ \\
\therefore D &= F_{\mathrm{BD}} = 291.7 \text{ N} \quad \measuredangle 36.87^\circ
\end{aligned}
$$

$$
\begin{aligned}
(+\rightarrow) \sum F_x &= 0 \\
&= A_x - F_{\mathrm{BD}} \cos\theta = 0 \\
&= A_x - (-291.7 \cos 36.87^\circ) = 0 \\
\therefore A_x &= -233.4 \text{ N} \\
&= 233.4 \text{ N} \quad (\leftarrow)
\end{aligned}
$$

$$
\begin{aligned}
(+\uparrow) \sum F_y &= 0 \\
&= A_y - 100 + F_{\mathrm{BD}} \sin\theta = 0 \\
&= A_y - 100 - (-291.7 \sin 36.87^\circ) = 0 \\
\therefore A_y &= -75 \text{ N} \\
&= 75 \text{ N} \quad (\downarrow)
\end{aligned}
$$

Solve for reaction at A;



$$
\begin{aligned}
\beta &= \tan^{-1} \frac{A_y}{A_x} = \frac{75}{233.4} = 17.8^\circ \\
A &= \sqrt{A_x^2 + A_y^2} = \sqrt{233.4^2 + 75^2} = 245.2 \text{ N} \\
\therefore A &= 245.2 \text{ N} \quad \measuredangle 17.8^\circ
\end{aligned}
$$

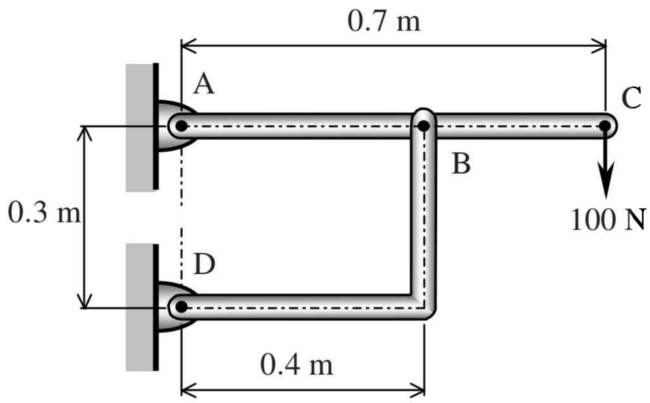
9.5 Three force member

For a three force member, the lines of action of the three forces must be either concurrent or parallel. Solution strategy is shown by following example.

A three force member is a rigid body of **negligible mass** with **only three forces** acting on it.

■ **Example 9.9**

Determine the reactions at A and D of the rigid body shown in the diagram.

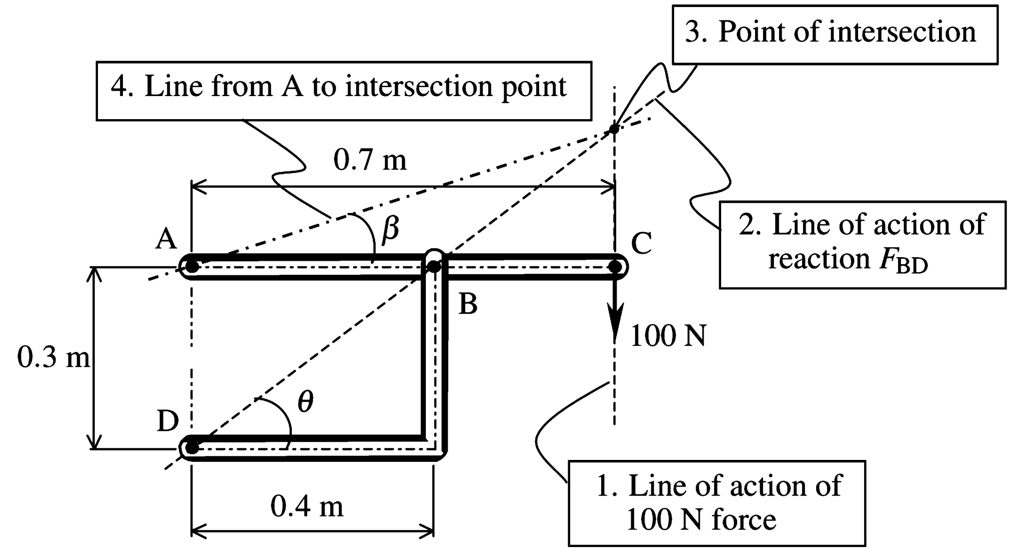


Solution:

It is important to ensure that there are only three forces acting on the body; the 100 N force and reactions at A and D. Notice that BD is a two force member, i.e. the line of action is known.

Step 1: Line of action

Draw the FBD and construct the two known lines of action. This will produce an intersection point (if this does not happen, then all three forces are parallel). The line of action of the third force is the line constructed from the third point to the intersection.

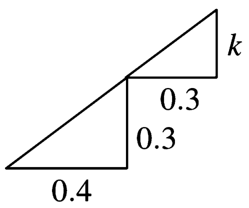


Step 2: Determine angles θ and β

$$\theta = \tan^{-1}\left(\frac{0.3}{0.4}\right) = 38.67^\circ$$

$$\beta = \tan^{-1}\left(\frac{k}{0.7}\right)$$

$$= \tan^{-1}\left(\frac{0.225}{0.7}\right) = 17.82^\circ$$

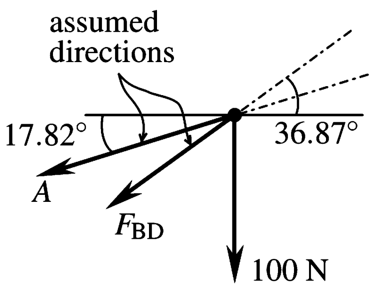


$$\frac{0.3}{0.4} = \frac{k}{0.3}$$

$$\therefore k = 0.225$$

Step 3: Determine magnitude of unknown forces

Separate the point of intersection and put in the known magnitude. The other two magnitudes can be found by solving using equilibrium of particles at the point of intersection.



$$(+\rightarrow)\sum F_x = 0$$

$$= -A\cos 17.82^\circ - F_{BD}\cos 36.87^\circ$$

$$\therefore A = -0.84F_{BD}$$

$$(+\uparrow)\sum F_y = 0$$

$$= -100 - F_{BD}\sin 36.87^\circ - A\sin 17.82^\circ$$

$$= -100 - F_{BD}\sin 36.87^\circ - (-0.84F_{BD})\sin 17.82^\circ$$

$$\therefore F_{BD} = -291.6\ \text{N}$$

$$\therefore A = -0.84F_{BD} = -0.84(-291.6) = 245\ \text{N}$$

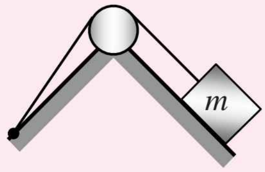
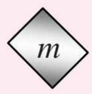
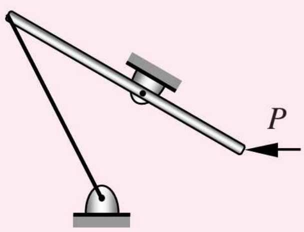
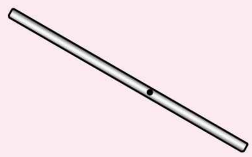
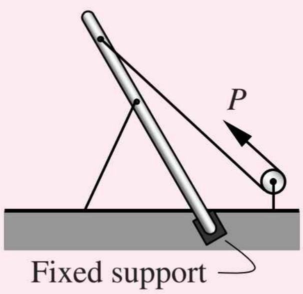
Therefore,

$$\therefore F_{BD} = 291.6\ \text{N} \quad \measuredangle 36.87^\circ$$
$$\therefore A = 245\ \text{N} \quad \overline{\nabla} 17.82^\circ$$

9.6 Example questions

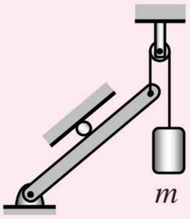
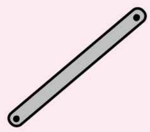
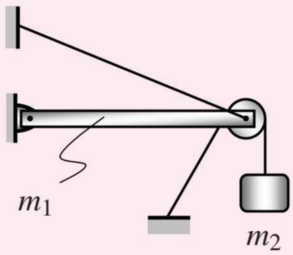
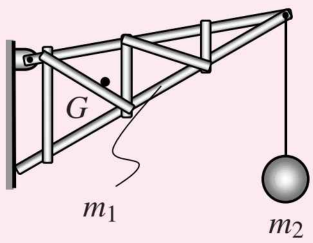
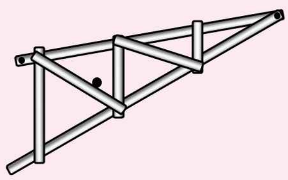
Exercise 9.1

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m, m_1 and m_2. All contacting surfaces are smooth unless stated otherwise.

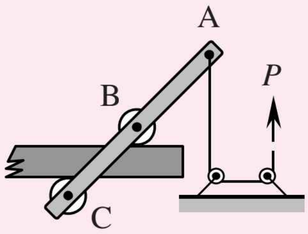
Problems	Free Body Diagrams
	
	
	

Exercise 9.2

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m, m_1 and m_2. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
	
	

Exercise 9.3
Complete the free body diagrams. Mass of the bodies is negligible unless stated by m, m_1 and m_2. All contacting surfaces are smooth unless stated otherwise.

<table>
<tr><th>Problems</th><th>Free Body Diagrams</th></tr>
<tr><td>


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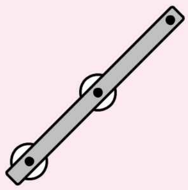
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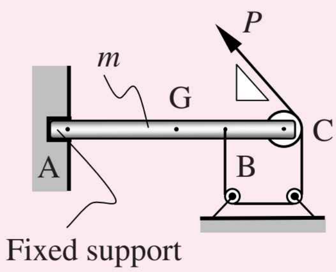

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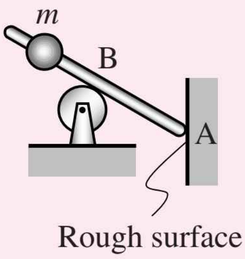

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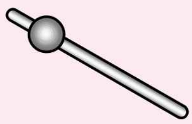
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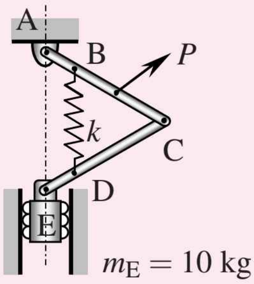
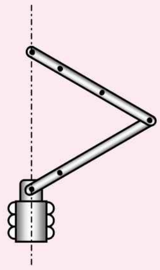
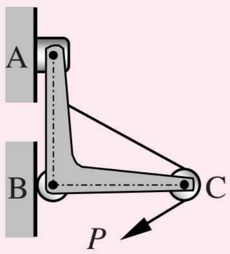
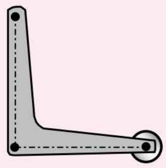
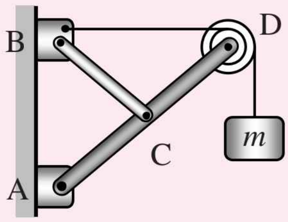
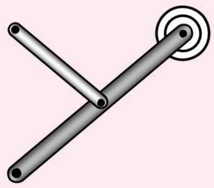
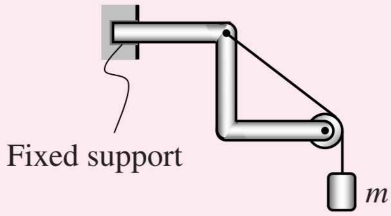
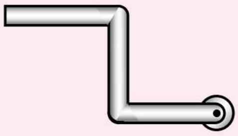
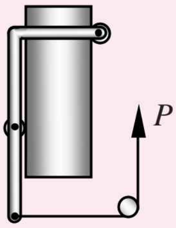
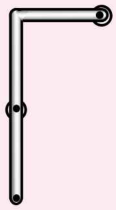
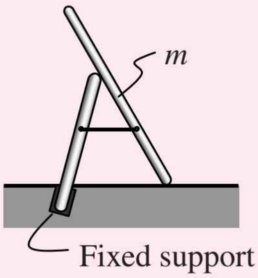
Exercise 9.4

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m, m_1 and m_2. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
	
	

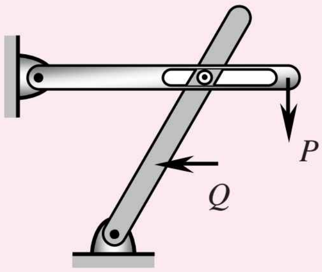
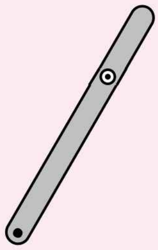
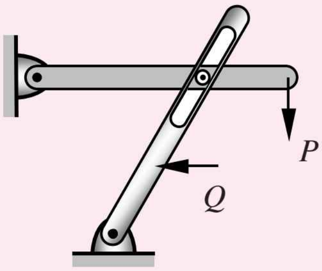
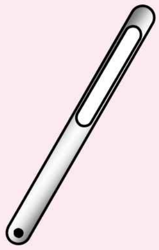
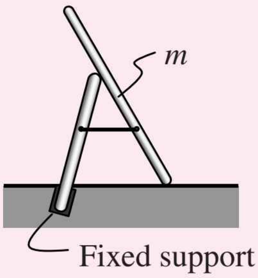
Exercise 9.5

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m, m_1 and m_2. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
	
	

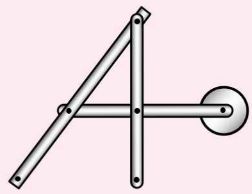
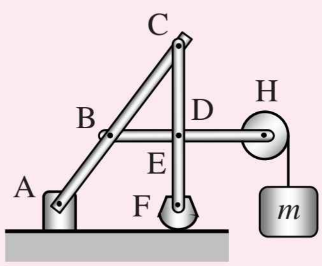
Exercise 9.6

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m, m_1 and m_2. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
	
	

Exercise 9.7

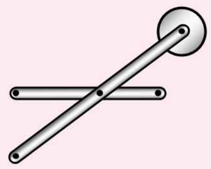
Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

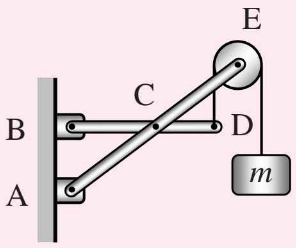
Problems	Free Body Diagrams
	
	
	

Exercise 9.8

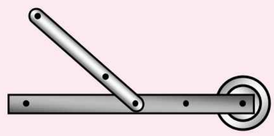
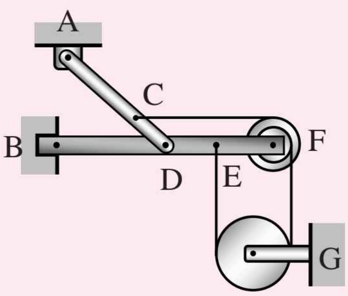
Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams



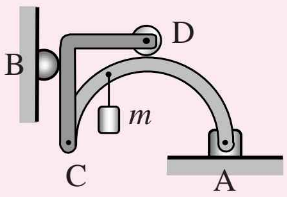
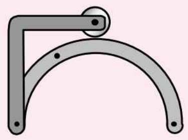
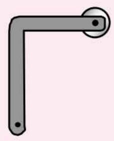
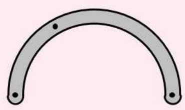


Exercise 9.9
Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
	
	

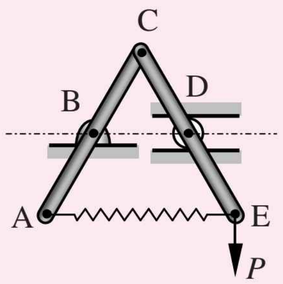
Exercise 9.10

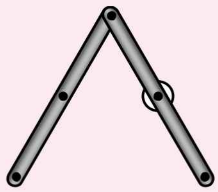
Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

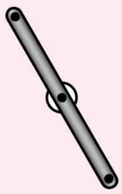
Problems	**Free Body Diagrams**
	
	
	

Exercise 9.11
Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems | **Free Body Diagrams**

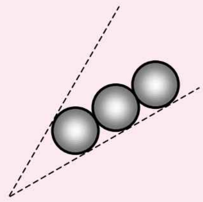
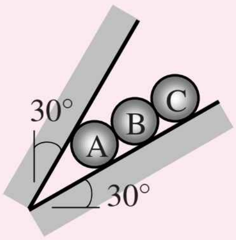
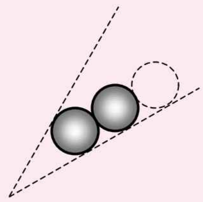
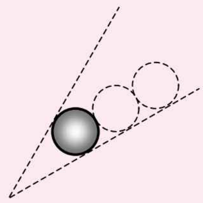






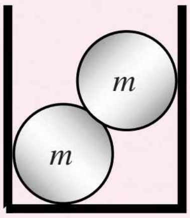
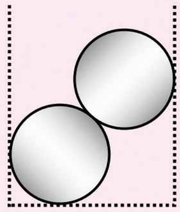
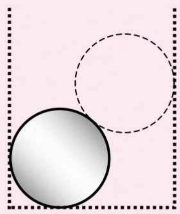
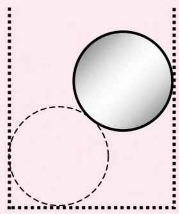
Exercise 9.12

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
$m_A = m_B = m_C = m$ 	
	

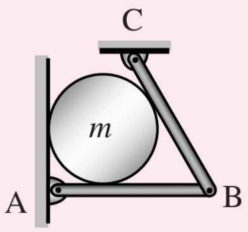
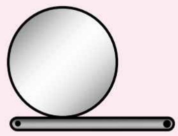
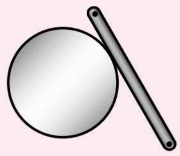
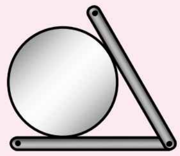
Exercise 9.13

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	  

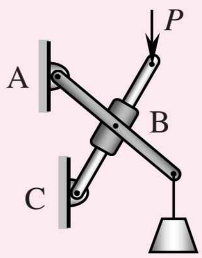
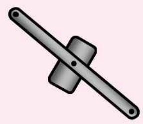
Exercise 9.14

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	  

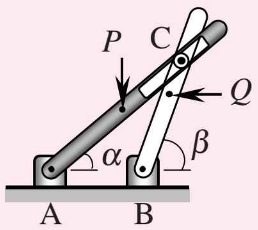
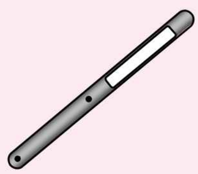
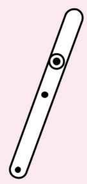
Exercise 9.15

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
 P A B C	
	

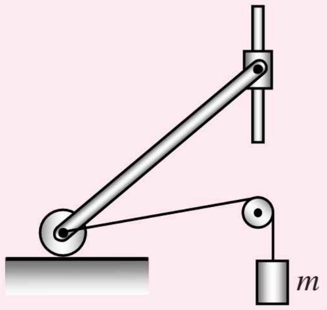
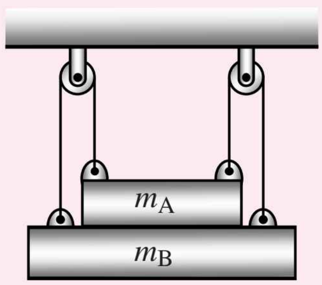
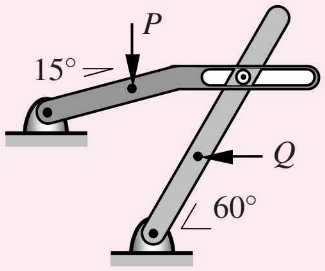
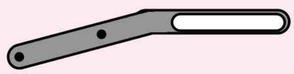
Exercise 9.16

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
	
	

Exercise 9.17

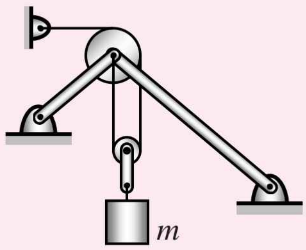
Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

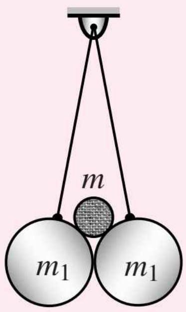
Problems	Free Body Diagrams
	
	
	

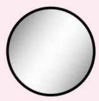
Exercise 9.18

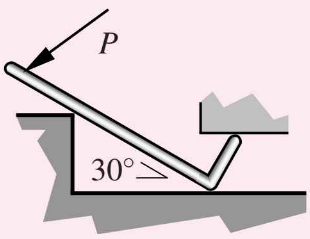
Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

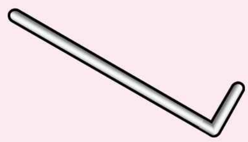
Problems	Free Body Diagrams



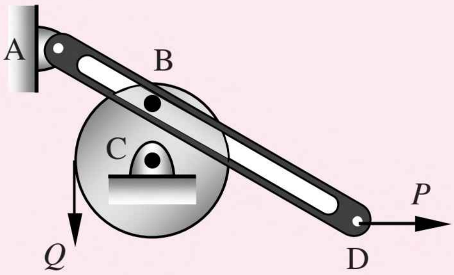
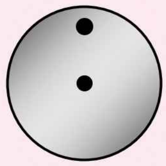
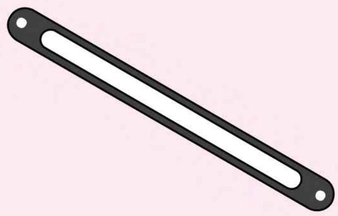
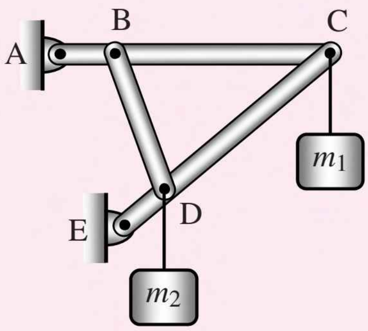








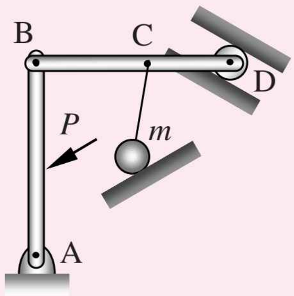
Exercise 9.19
Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

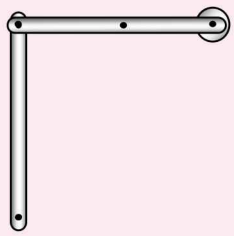
Problems	Free Body Diagrams
	 
	 

Exercise 9.20

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	**Free Body Diagrams**

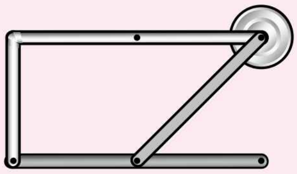


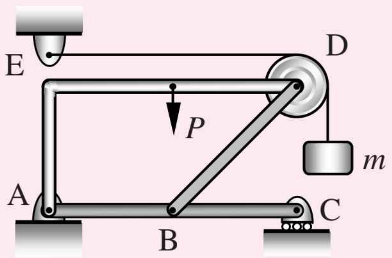


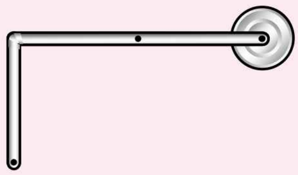
Exercise 9.21

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams

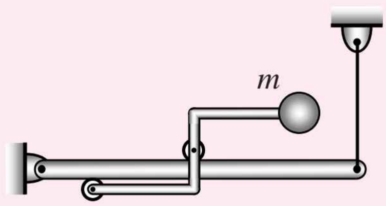
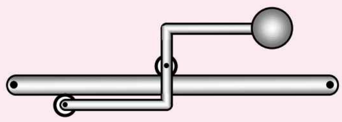
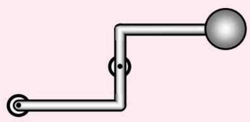






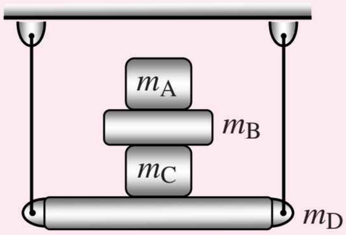
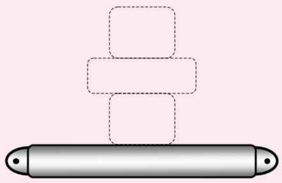
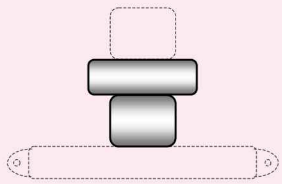
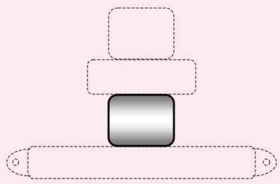
Exercise 9.22

Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	  

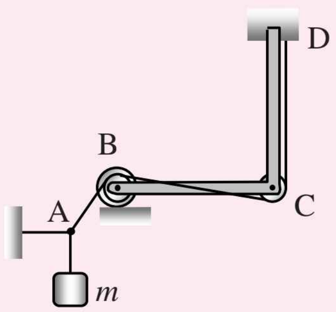
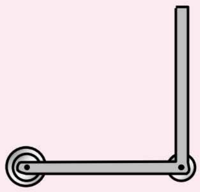
Exercise 9.23

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	  

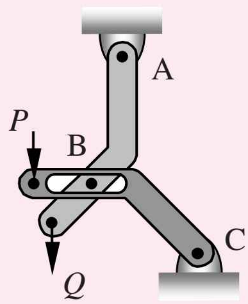
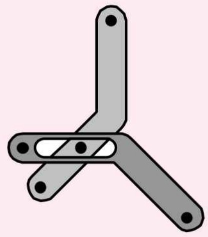
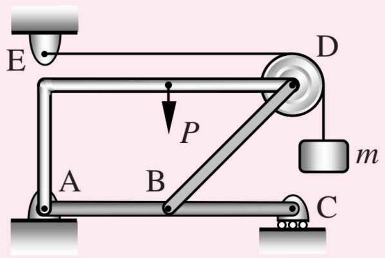
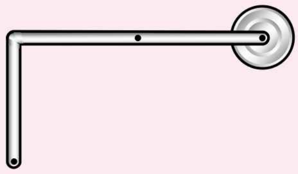
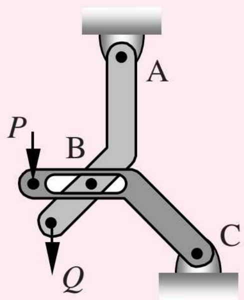
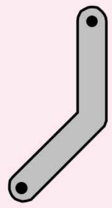
Exercise 9.24

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	
	
	

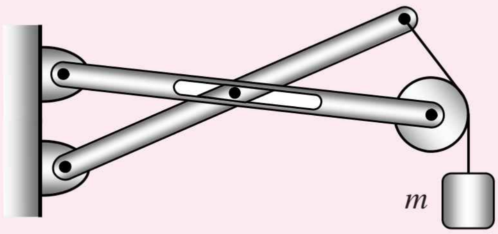
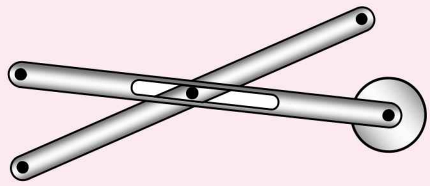
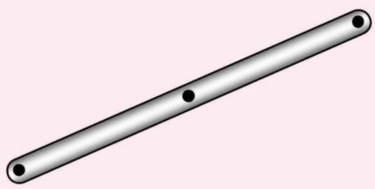
Exercise 9.25

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

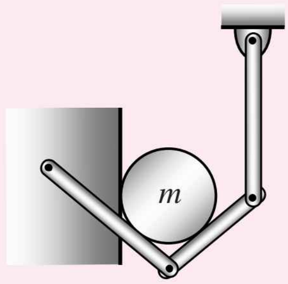
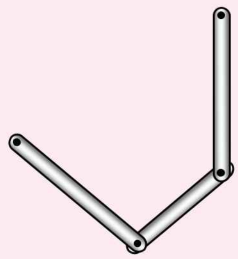
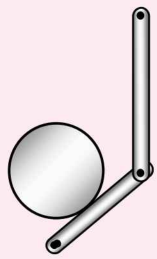
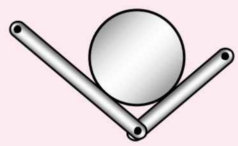
Problems	Free Body Diagrams
	
	
	

Exercise 9.26

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

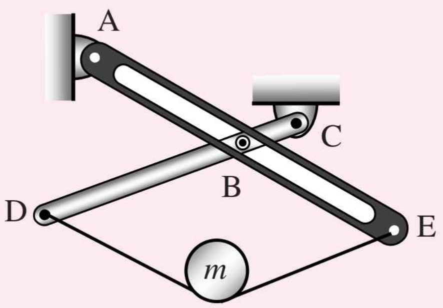
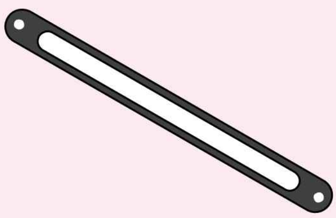
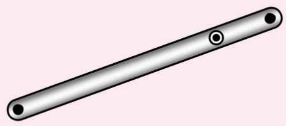
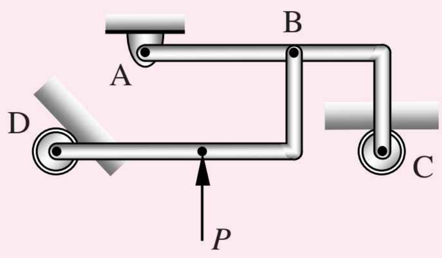
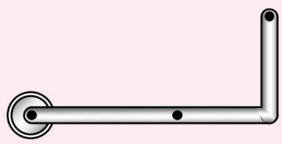
Problems	**Free Body Diagrams**
	  

Exercise 9.27
Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	  

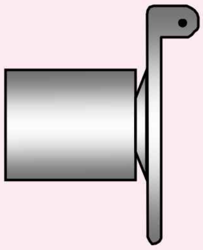
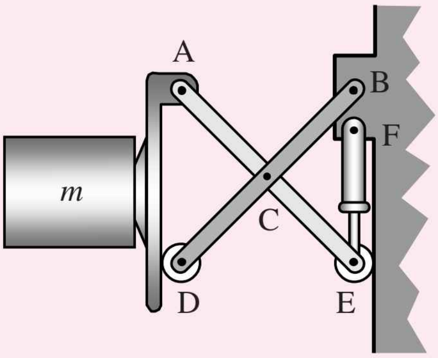
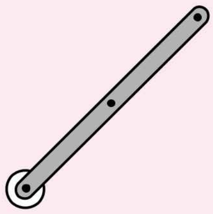
Exercise 9.28

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	 
	 

Exercise 9.29

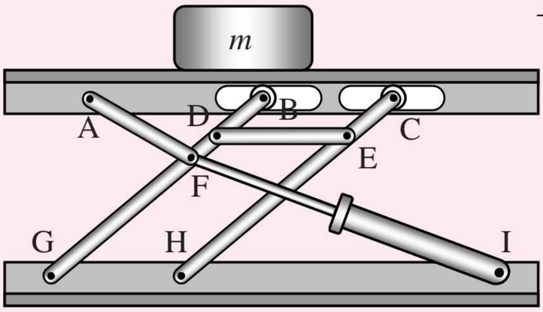
Complete the free body diagrams. Mass of the bodies is negligible unless stated by m. All contacting surfaces are smooth unless stated otherwise.

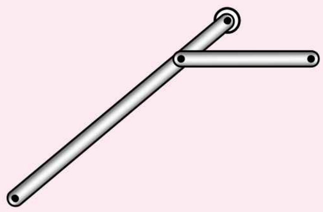
Problems	**Free Body Diagrams**
	
	
	

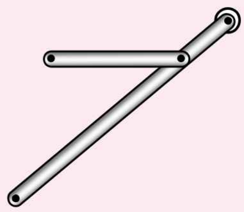
Exercise 9.30

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	**Free Body Diagrams**

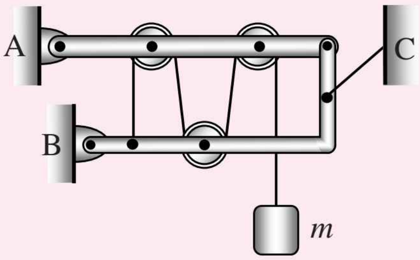
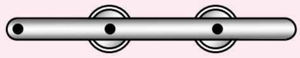
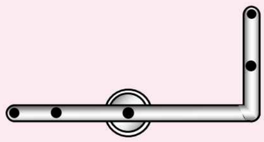






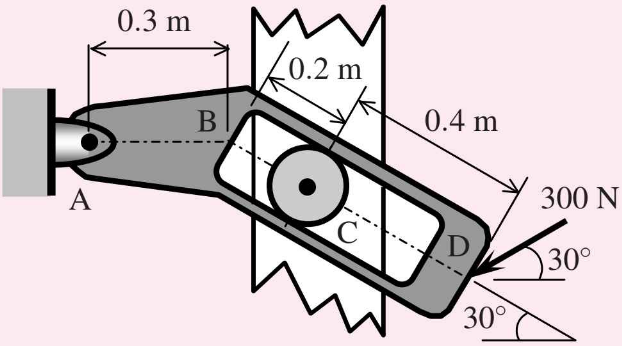
Exercise 9.31

Complete the free body diagrams. Mass of the bodies is negligible unless stated by *m*. All contacting surfaces are smooth unless stated otherwise.

Problems	Free Body Diagrams
	 

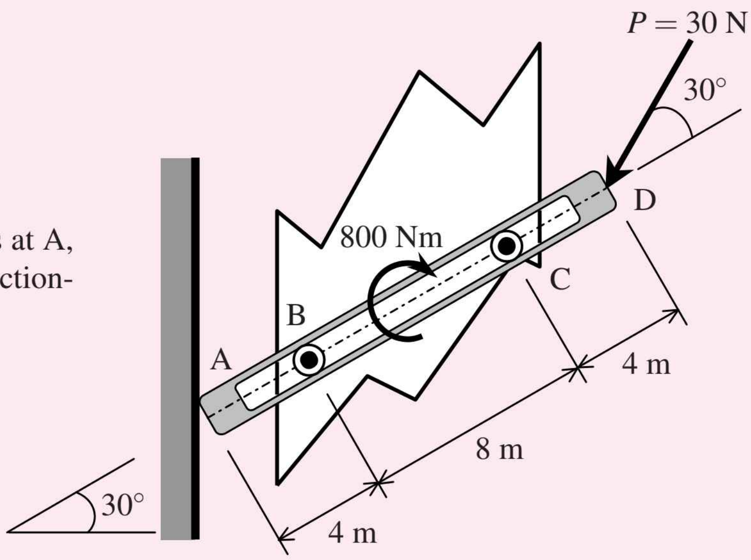
Exercise 9.32

A slot has been cut in plate ABD. Determine the reactions at A and C.
Answer: $R_C = 436.5$ N (60° ∡) and $R_A = 232$ N (79.7° ⦪)



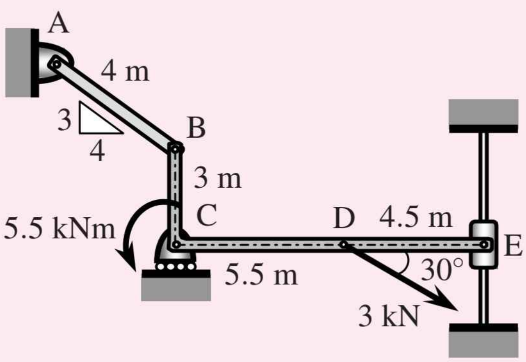
Exercise 9.33

Determine the reactions at A, B and C. The slot is frictionless.



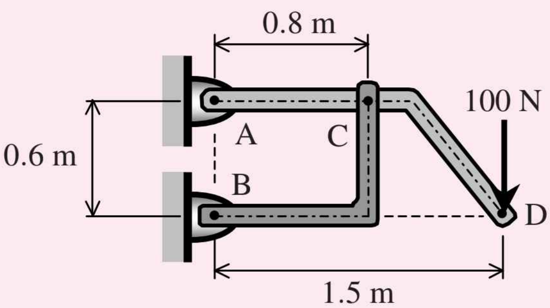
Exercise 9.34

Bent rod BCDE is supported by a roller at C, a smooth collar at E and a light arm AB at B. Determine the reactions at B, C and E for the bent rod BCDE.



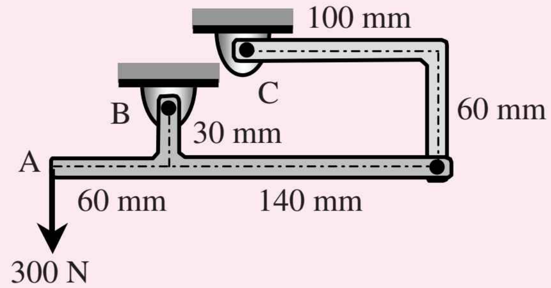
Exercise 9.35

Determine the reactions at A and B.
Answer: $R_B = 312.5$ N (36.87° ∡) and $R_A = 264.9$ N (19.3° ∡)



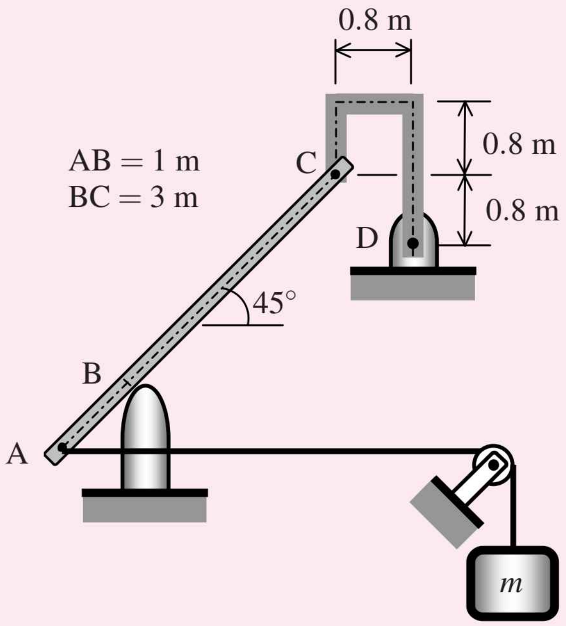
Exercise 9.36

Determine the reactions at B and C for the system when a 300 N force is exerted at A.



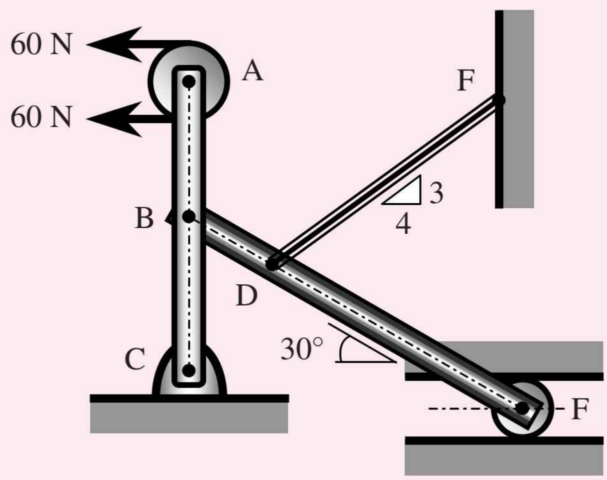
Exercise 9.37

The diagram shows rigid body ABC with a mass of 10 kg supported by member CD at C and loaded with a mass m at end A. Determine the mass m and reactions at B and C so that the system is in equilibrium.
Answer: $R_C = 0.0057$ N, $R_B = 138.7$ N (45° ∡) and $m = 10$ kg



Exercise 9.38

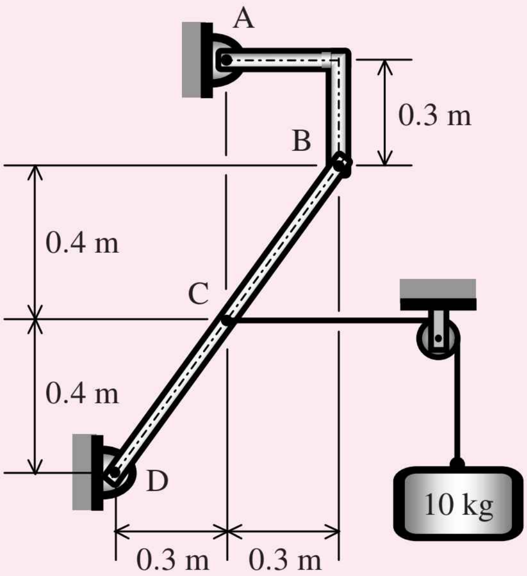
The diagram shows a mechanism used to support the load at the pulley. The tension in cable DF is 50 N. Determine the reactions at C and E if the system is in equilibrium. All contacting surfaces are frictionless.



AB = 70 mm   BD = 50 mm
BC = 80 mm   DE = 150 mm

Radius of pulley at A = 40 mm.

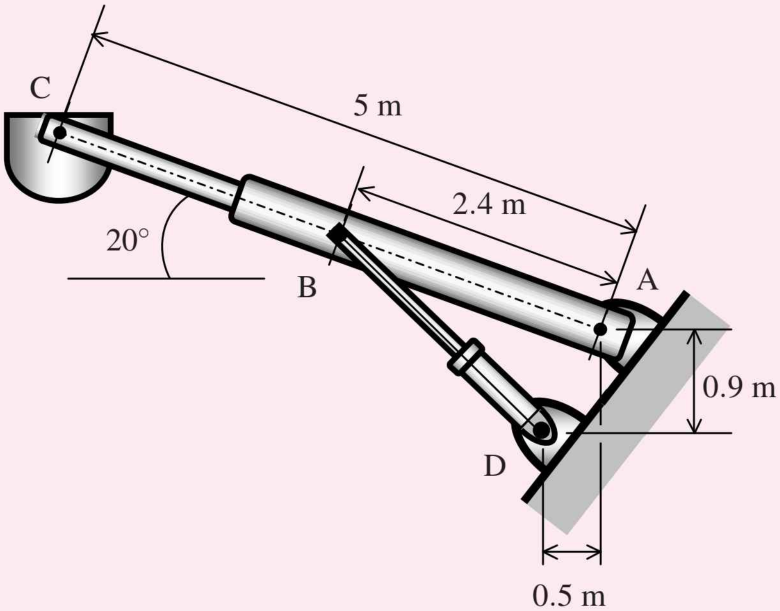
Exercise 9.39



The system is in equilibrium. Determine the reactions at A and D if the mechanism is loaded as shown in the diagram.
Answer: $R_A = 39.6$ N (45° ⦣) and $R_D = 75.5$ N (21.8° ⦣)

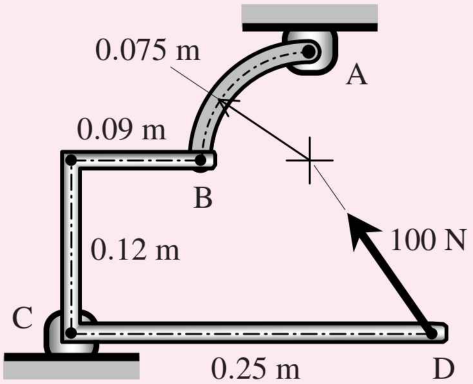
Exercise 9.40

Telescopic arm ABC is used to support a 200 kg load at C. Determine the force in hydraulic cylinder BD and the reaction at A.



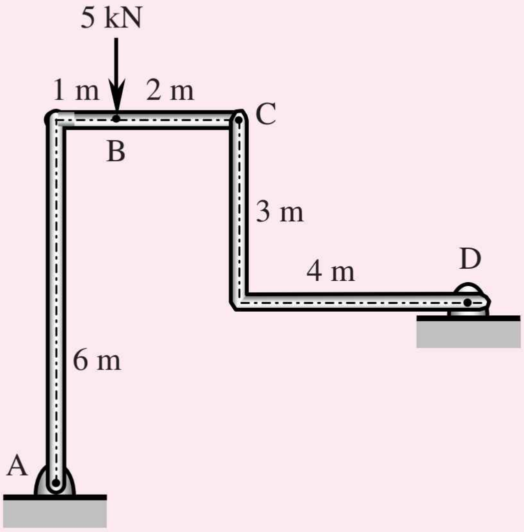
Exercise 9.41

Determine the reactions at A and C of the mechanism shown in the diagram.



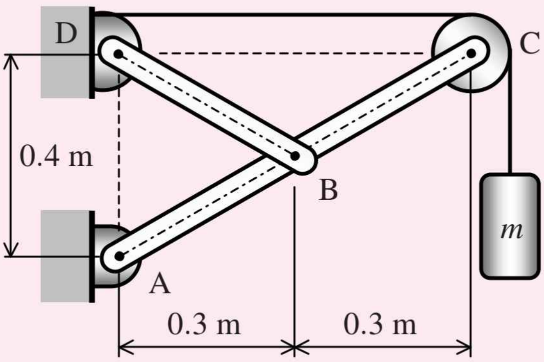
Exercise 9.42

Determine the reaction at pins A and D of the structure shown in the figure when a 5 kN force is acted at B. Answer: $R_D = 757.6$ N (36.87° ∡) and $R_A = 4585.7$ N (82.4° ∡)



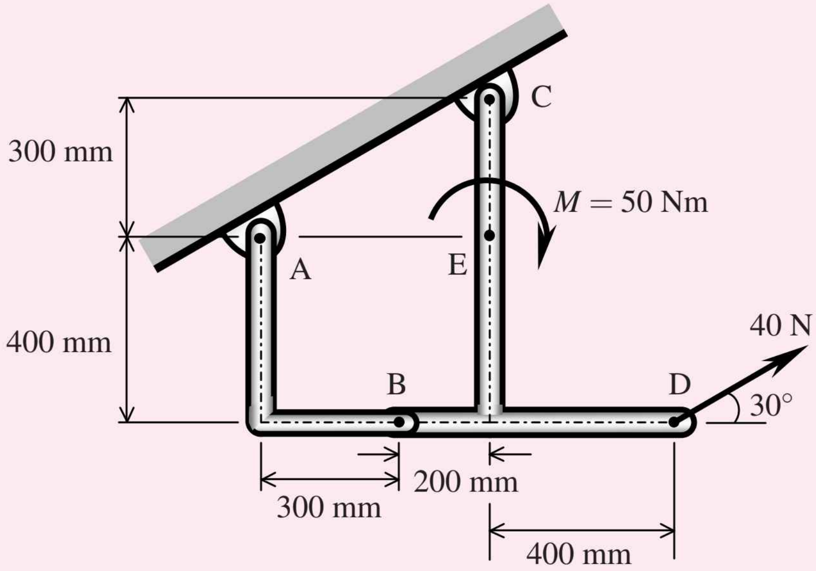
Exercise 9.43

Determine the tension in the cable and the reactions at pins A and D. The 0.1 m diameter pulley is frictionless and the mass $m = 100$ kg.



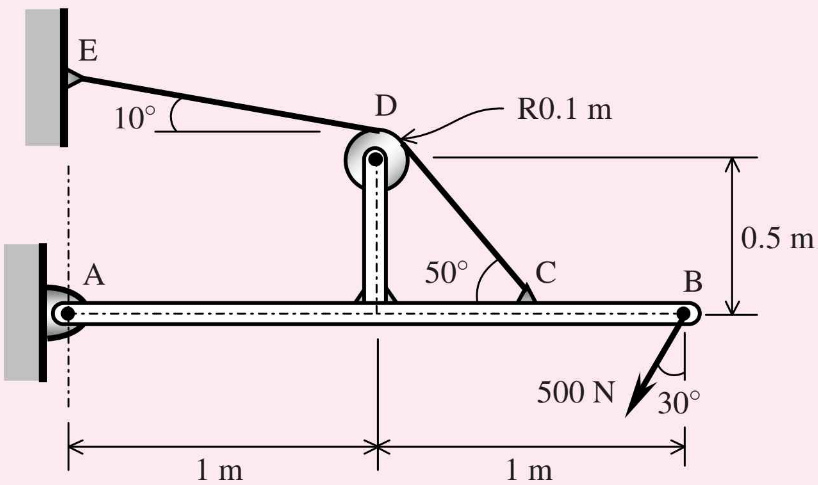
Exercise 9.44

Determine the reactions at A and C resulting from the 40 N force at D and the 50 Nm couple at E.



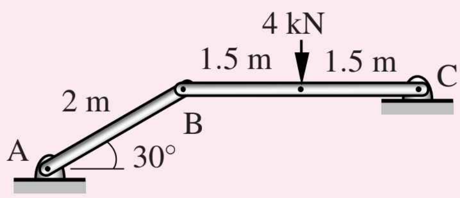
Exercise 9.45

The rigid body shown is supporting the 500 N load at B. Determine the reaction at A and the tension in cable EDC. The system is in equilibrium.



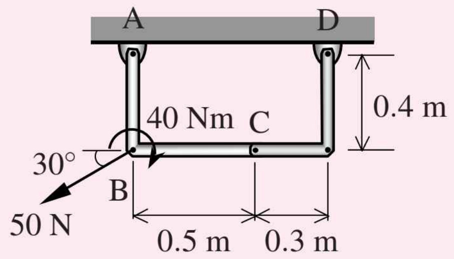
Exercise 9.46

Determine the reactions at pin A and C to support the 4 kN load. Neglect the mass of rods AB and BC.



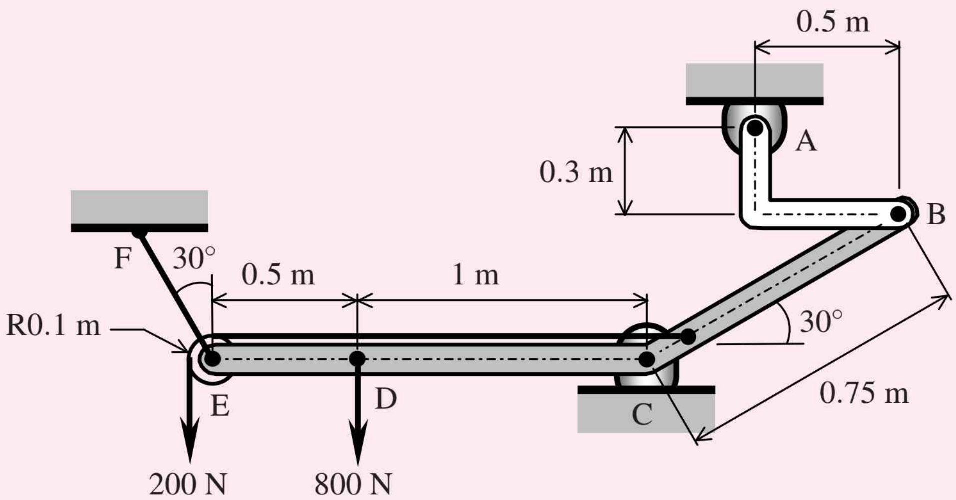
Exercise 9.47

Determine the reactions at A and D of the mechanism shown.



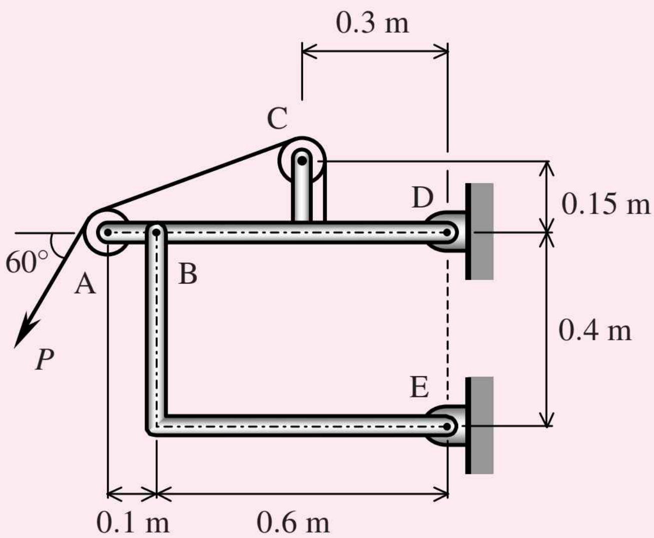
Exercise 9.48

Determine the reactions at C and A if the tension in cable EF is 200 N.



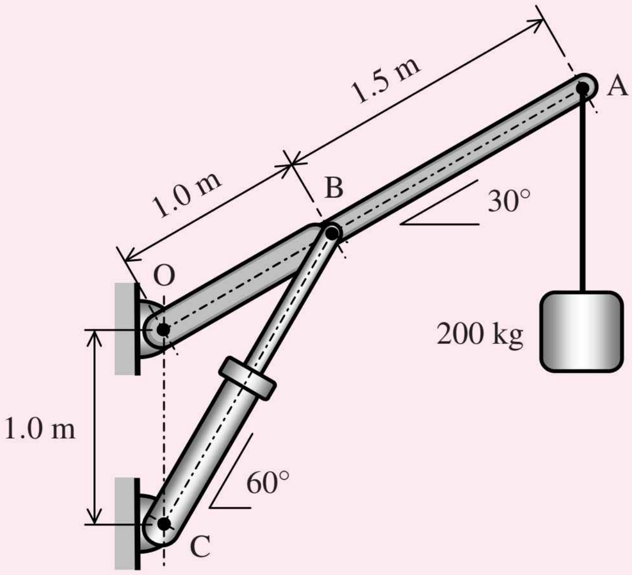
Exercise 9.49

Determine the reaction at pins D and E for the structure shown. The force $P = 100$ N and diameter of both pulleys is 0.1 m.



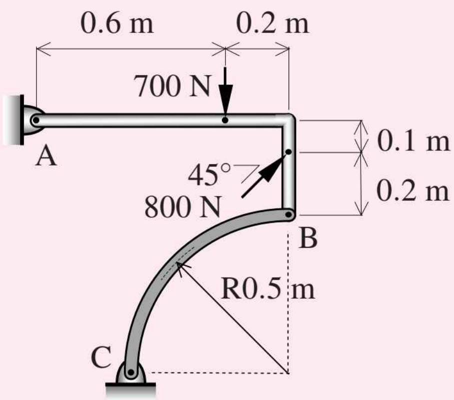
Exercise 9.50

Determine the reaction at pin O and the force in hydraulic cylinder BC for the crane in the figure.



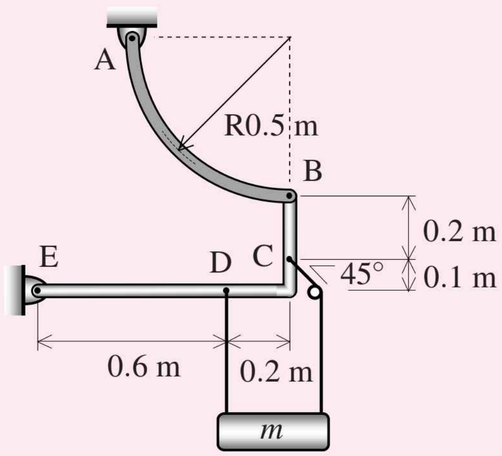
Exercise 9.51

Determine the reactions at A and C if the system is in equilibrium.



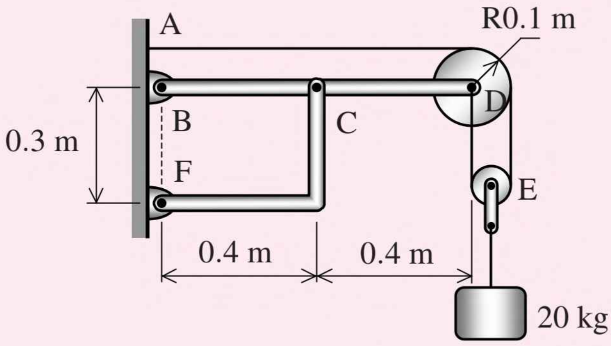
Exercise 9.52

The system shown is in equilibrium. Determine the reaction at A and E if $m = 20$ kg. Radius of the small pulley can be neglected.



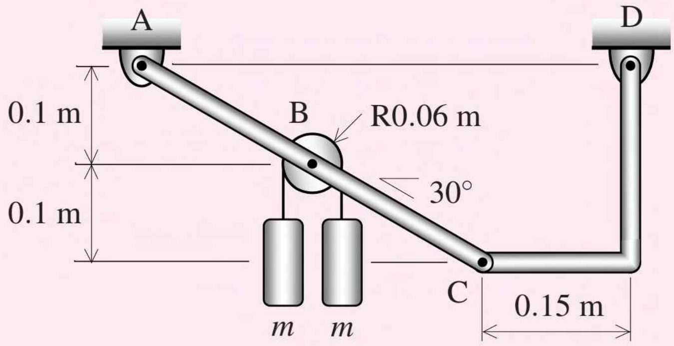
Exercise 9.53

The mechanism shown is used to support the 20 kg mass. Determine the reaction at pins B and F.



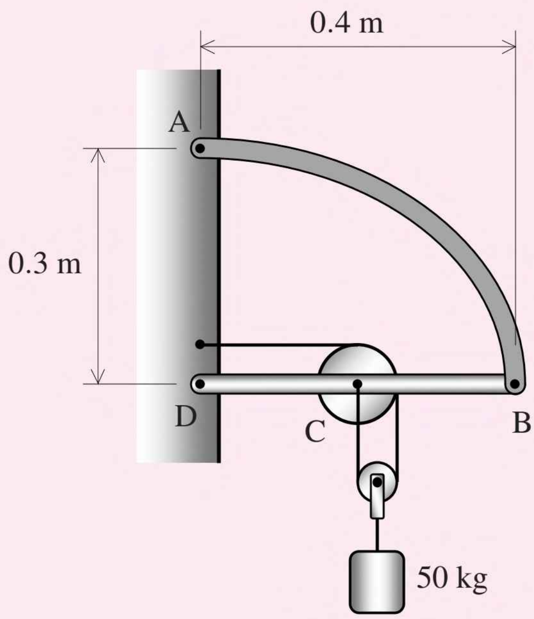
Exercise 9.54

The system shown is in equilibrium. Determine the reaction at pins A and D if $m = 10$ kg.



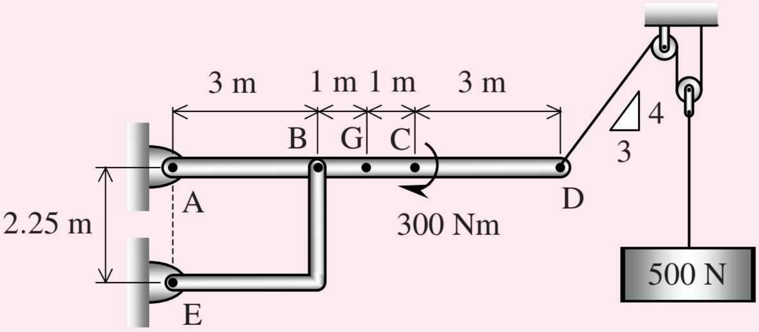
Exercise 9.55

Determine the reaction at A and D if the system is in equilibrium. The big pulley is at the mid point of member BD. The big pulley diameter is 0.1 m while the small pulley diameter is 0.05 m.



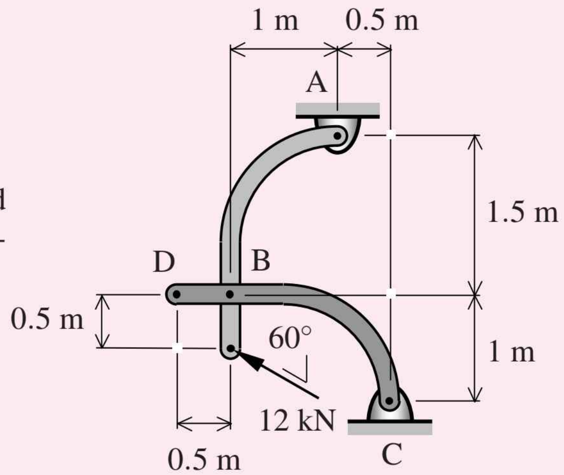
Exercise 9.56

Rigid body ABCD of weight 100 N (acting at G) is used to support the 500 N load and the 300 Nm couple. Determine the reaction at A and E.



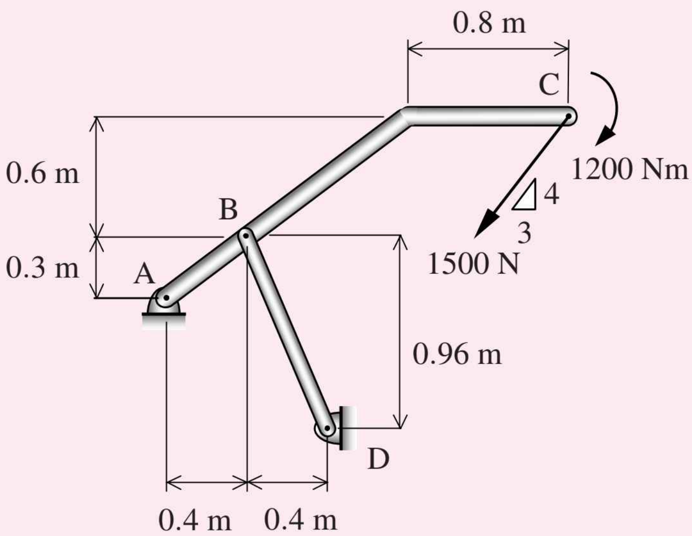
Exercise 9.57

Determine the reaction at pins A and C for the system to maintain equilibrium.



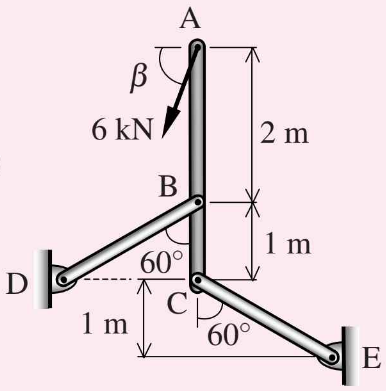
Exercise 9.58

Determine the reaction at pins A and D for the system to maintain equilibrium.



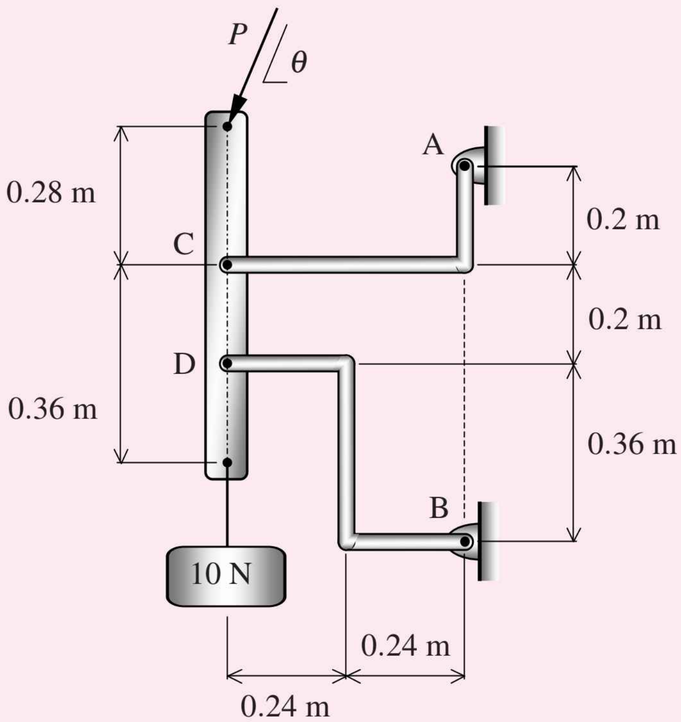
Exercise 9.59

Determine the reaction at B and C, and the angle β to maintain ABC in the vertical position.



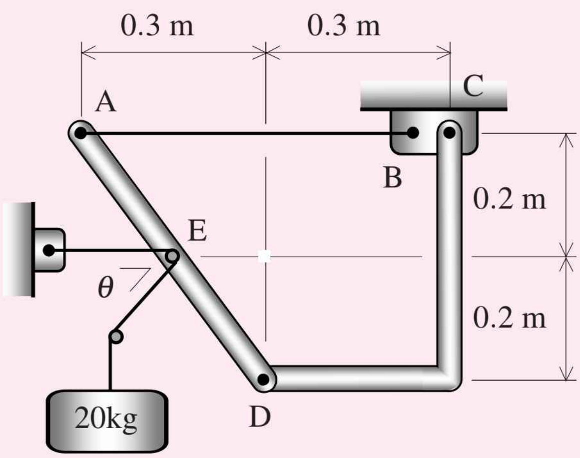
Exercise 9.60

The system shown is in equilibrium. Determine the force P, the angle θ and the reaction at pin A if the compression in member BD is 50 N.



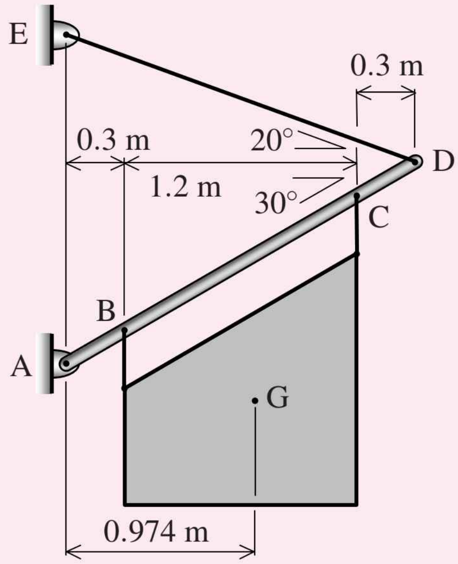
Exercise 9.61

Determine the angle θ, tension in cable AB and reaction at C for the system to be in equilibrium. The pulley radius can be neglected.



Exercise 9.62

Determine the tension in cables at B and C of the hanging 10 kg signboard (using FBD of signboard). Hence, determine the tension in cable DE and the reaction at A (using overall FBD).



10 — Rigid bodies : Equilibrium 3D

10.1 Equilibrium condition in 3D

The differences between equilibrium conditions of particle and rigid body in three dimension (i.e. x-y-z plane) are shown in the following table;

	Equilibrium conditions	Notes
Particles	$\sum F_x = 0$ and $\sum F_y = 0$ and $\sum F_z = 0$	3 equations up to 3 unknowns
Rigid bodies	$\sum F_x = 0$ and $\sum F_y = 0$ and $\sum F_z = 0$ $\sum M_x = 0$ and $\sum M_y = 0$ and $\sum M_z = 0$	6 equations up to 6 unknowns

For solving the equilibrium conditions, free body diagram (FBD) of the rigid body is needed to identify the relevant forces and moments.

10.2 FBD for 3D rigid bodies

Procedure for drawing FBD is;

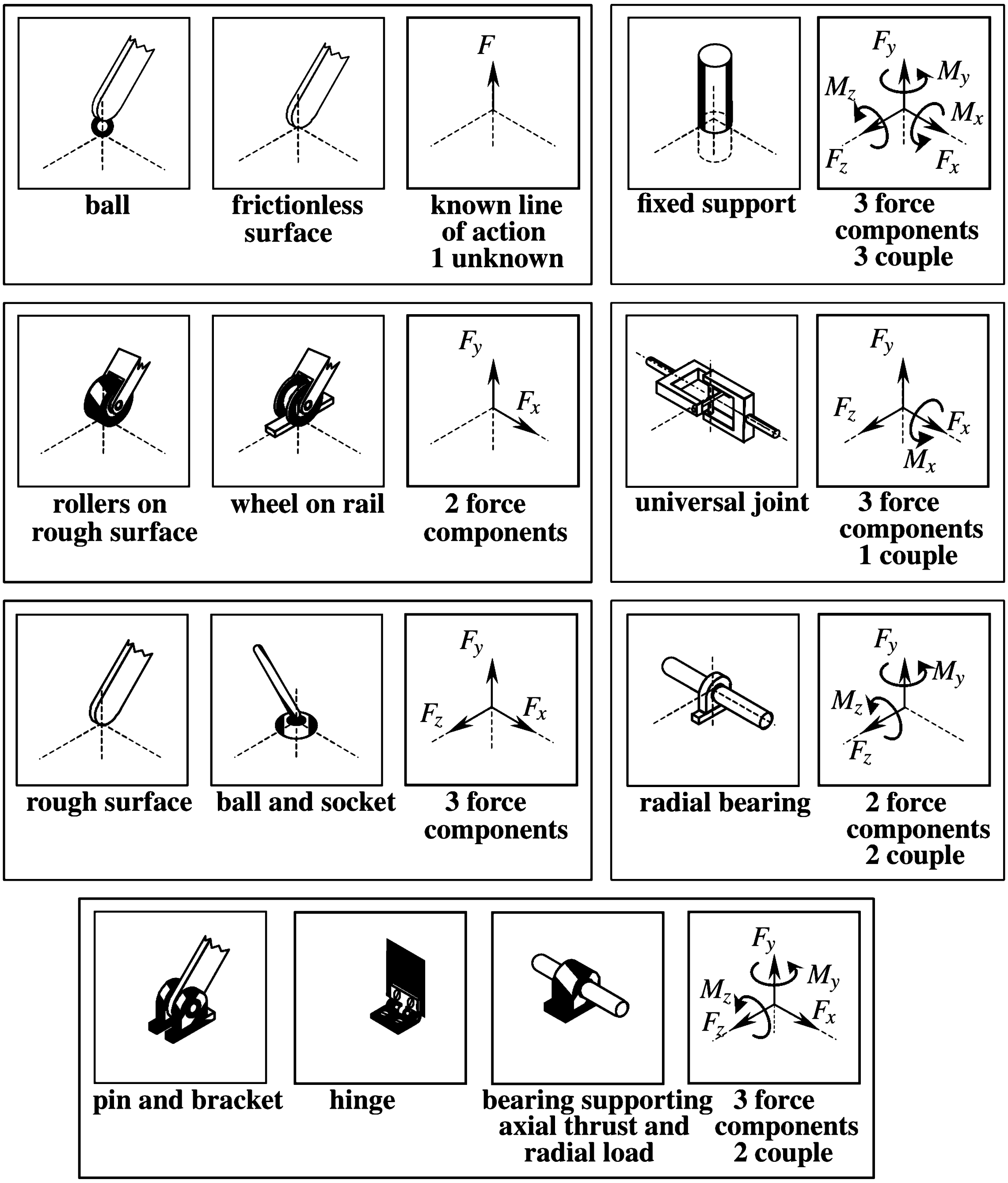
- Draw the boundary of the chosen section and detach / separate it from all other bodies,
- Input all external forces acting on the body,
- For body with a mass, put the weight $= mg$ acting at the centre of gravity (G) of the body in the vertically downwards direction,
- Put the reaction/s where the body touches or connected to a different rigid body,
- The FBD should also include dimensions for the process of taking moment.

When supports or connections on the rigid body are removed, they are replaced with reaction forces and/or reaction moments. These information are summarized in Tables 10.1 and 10.2.

Table 10.1: Reaction forces / moments for respective supports / connections in 3D

Supports / Connections	**Reaction forces / moments**	**Notes on reaction forces / moments**
• Balls • Frictionless surface	A force with a known line of action involving 1 unknown	Perpendicular to the surface and must point towards the free body
• Roller on rough surface • Wheel on rail	Two force components, involving 2 unknowns	One perpendicular to the surface and must point towards the free body and the other 90° to it (tangent to the surface) that can be directed either way (not both), preventing translation in two directions
• Rough surface • Ball and socket	Three force components, involving 3 unknowns	The force components are usually represented by their x, y and z components, preventing translation in three directions
• Universal joint	Three force components and one couple, involving 4 unknowns	The force components prevent translation in three directions. The universal joint is designed to allow rotation about two axes
• Fixed support • Welded joint	Three force components and three couples, involving 6 unknowns	The force components prevent translation in three directions. The couple components prevent rotation about three axes
• Hinge • Bearing supporting radial load only	Two force components (and two couples), involving 2 (or 4) unknowns	Mainly designed to prevent translation in two directions, that may also include two couples. Generally will not exert any appreciable couple unless otherwise stated
• Pin and bracket • Hinge • Bearing supporting axial thrust and radial load	Three force components (and two couples), involving 3 (or 5) unknowns	Designed to prevent translation in three directions, which may also include two couples. Generally will not exert any appreciable couple unless stated otherwise

Table 10.2: Summary of reaction forces / moments in 3D

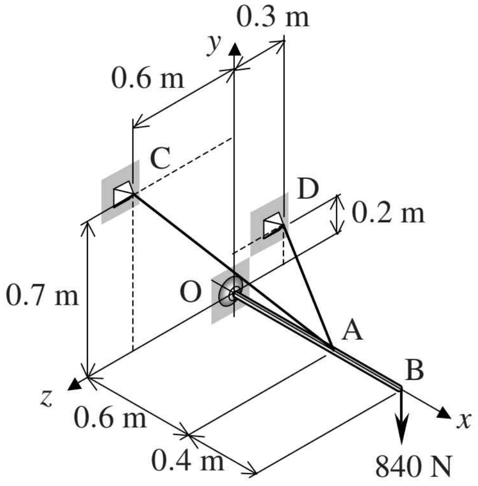


Important notes

- The FBD must be drawn for every solution.
- Knowledge on determination of force components, reactions and 'cross and dot products' (for taking moments) are essential.
- Since there are 6 equations, there can be up to 6 unknowns. For problems with more unknowns, solve using moment about an axis.

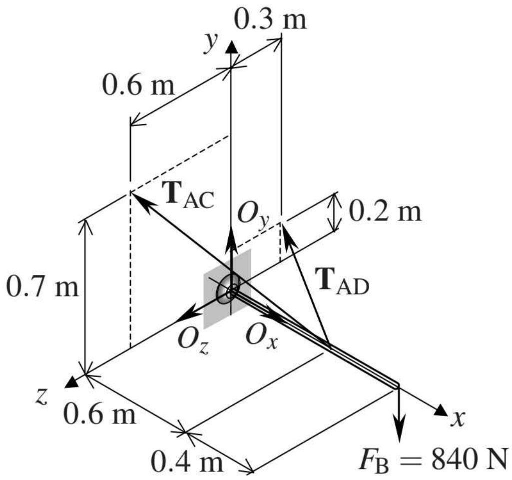
▪ Example 10.1

The system shown in the diagram is in equilibrium. Determine the tension in cables AC and AD, and the reaction at the ball and socket joint at O.



Solution:

1. FBD



2. Solving components of the forces

Force applied at point B, $\mathbf{F}_B = -840 \text{ N } \mathbf{j}$

Tension in the cable AC, $\mathbf{T}_{\text{AC}}$;

$$d_x = -0.6 \text{ m}$$
$$d_y = 0.7 \text{ m}$$
$$d_z = 0.6 \text{ m}$$
$$\therefore d = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{(-0.6)^2 + (0.7)^2 + (0.6)^2} = 1.1 \text{ m}$$

$$T_{\text{ACx}} = T_{\text{AC}}\frac{d_x}{d} = T_{\text{AC}}\frac{(-0.6)}{1.1} = -0.545T_{\text{AC}}$$
$$T_{\text{ACy}} = T_{\text{AC}}\frac{d_y}{d} = T_{\text{AC}}\frac{(0.7)}{1.1} = 0.636T_{\text{AC}}$$
$$T_{\text{ACz}} = T_{\text{AC}}\frac{d_z}{d} = T_{\text{AC}}\frac{(0.6)}{1.1} = 0.545T_{\text{AC}}$$
$$\therefore \mathbf{T}_{\text{AC}} = -0.545T_{\text{AC}} \text{ N } \mathbf{i} + 0.636T_{\text{AC}} \text{ N } \mathbf{j} + 0.545T_{\text{AC}} \text{ N } \mathbf{k}$$

Tension in the cable AD, $\mathbf{T}_{\text{AD}}$;

$$d_x = -0.6 \text{ m}$$
$$d_y = 0.2 \text{ m}$$
$$d_z = -0.3 \text{ m}$$
$$\therefore d = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{(-0.6)^2 + (0.2)^2 + (-0.3)^2} = 0.7 \text{ m}$$

$$T_{\text{ADx}} = T_{\text{AD}}\frac{d_x}{d} = T_{\text{AD}}\frac{(-0.6)}{0.7} = -0.857T_{\text{AD}}$$
$$T_{\text{ADy}} = T_{\text{AD}}\frac{d_y}{d} = T_{\text{AD}}\frac{(0.2)}{0.7} = 0.286T_{\text{AD}}$$
$$T_{\text{ADz}} = T_{\text{AD}}\frac{d_z}{d} = T_{\text{AD}}\frac{(-0.3)}{0.7} = -0.429T_{\text{AD}}$$
$$\therefore \mathbf{T}_{\text{AD}} = -0.857T_{\text{AD}} \text{ N } \mathbf{i} + 0.286T_{\text{AD}} \text{ N } \mathbf{j} - 0.429T_{\text{AD}} \text{ N } \mathbf{k}$$

Reaction at point O;

$$\mathbf{O} = O_x \text{ N } \mathbf{i} + O_y \text{ N } \mathbf{j} + O_z \text{ N } \mathbf{k}$$

Force	**i** component	**j** component	**k** component
$\mathbf{F}_{\text{B}}$		-840	
$\mathbf{T}_{\text{AC}}$	-0.545 T_{AC}	0.636 T_{AC}	0.545 T_{AC}
$\mathbf{T}_{\text{AD}}$	-0.857 T_{AD}	0.286 T_{AD}	-0.429 T_{AD}
$\mathbf{O}$	O_x	O_y	O_z
$\sum F$	0	0	0

For equilibrium $\sum F = 0$. By adding the forces according to respective components give three equations;

$$\mathbf{i} \text{ component} \Rightarrow -0.545T_{\text{AC}} - 0.857T_{\text{AD}} + O_x = 0 \quad \ldots(1)$$
$$\mathbf{j} \text{ component} \Rightarrow -840 + 0.636T_{\text{AC}} + 0.286T_{\text{AD}} + O_y = 0 \quad \ldots(2)$$
$$\mathbf{k} \text{ component} \Rightarrow 0.545T_{\text{AC}} - 0.429T_{\text{AD}} + O_z = 0 \quad \ldots(3)$$

3. Solving components of the moments

Taking moment about O (the intersection point with the most unknowns) gives;

$$\sum M_O = \mathbf{r}_{OB} \times (-840 \text{ N } \mathbf{j}) + \mathbf{r}_{OA} \times \mathbf{T}_{AC} + \mathbf{r}_{OA} \times \mathbf{T}_{AD}$$

moment of the $\mathbf{F}_B = 840 \text{ N } \mathbf{j}$ force about point O;

$$\begin{aligned}\sum M_O =& \mathbf{r}_{OB} \times (-840 \text{ N } \mathbf{j}), \qquad \text{where } \mathbf{r}_{OB} = 1 \text{ m } \mathbf{i}\\ =&(1 \text{ m } \mathbf{i}) \times (-840 \text{ N } \mathbf{j}) = -840 \text{ Nm } \mathbf{k}\end{aligned}$$

moment of the $\mathbf{T}_{AC}$ about point O;

$$\begin{aligned}\sum M_O =& \mathbf{r}_{OA} \times T_{AC}, \qquad \text{where } \mathbf{r}_{OA} = 0.6 \text{ m } \mathbf{i}\\ =&(0.6 \text{ m } \mathbf{i}) \times (-0.545 T_{AC} \text{ N } \mathbf{i}) + 0.636 T_{AC} \text{ N } \mathbf{j} + 0.545 T_{AC} \text{ N } \mathbf{k})\\ =&(0.6)(0.636 T_{AC})(+\mathbf{k}) + (0.6)(0.545 T_{AC})(-\mathbf{j})\\ =&-0.327 T_{AC} \mathbf{j} + 0.3816 T_{AD} \mathbf{k}\end{aligned}$$

moment of the $\mathbf{T}_{AD}$ about point O;

$$\begin{aligned}\sum M_O =& \mathbf{r}_{OA} \times \mathbf{T}_{AD}, \qquad \text{where } \mathbf{r}_{OA} = 0.6 \text{ m } \mathbf{i}\\ =&(0.6 \text{ m } \mathbf{i}) \times (-0.857 T_{AD} \text{ N } \mathbf{i}) + 0.286 T_{AD} \text{ N } \mathbf{j} - 0.429 T_{AD} \text{ N } \mathbf{k})\\ =&(0.6)(0.286 T_{AD})(+\mathbf{k}) + (0.6)(-0.429 T_{AD})(-\mathbf{j})\\ =&0.257 T_{AD} \mathbf{j} + 0.1716 T_{AD} \mathbf{k}\end{aligned}$$

Moment source	**i** component	**j** component	**k** component
$\mathbf{F}_B$			-840
$\mathbf{T}_{AC}$		-0.327 T_{AC}	0.3816 T_{AC}
$\mathbf{T}_{AD}$		0.257 T_{AD}	0.1716 T_{AD}
$\sum M$	0	0	0

For equilibrium $\sum M = 0$. By adding the moments according to respective components give additional two equations;

$$\begin{aligned}&\mathbf{j} \text{ component} \quad \Rightarrow \quad -0.327 T_{AC} + 0.257 T_{AD} = 0 \quad \ldots(4)\\ &\mathbf{k} \text{ component} \quad \Rightarrow \quad -840 + 0.381 T_{AC} + 0.1716 T_{AD} = 0 \quad \ldots(5)\end{aligned}$$

from equation (4),

$$T_{AC} = \frac{0.257}{0.327} T_{AD} = 0.786 T_{AD}$$

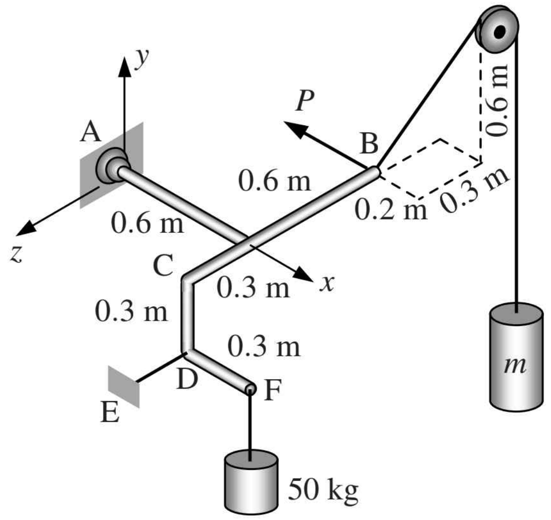
input into (5),

$$-840+0.3816(0.786T_{AD})+0.1716T_{AD}=0$$

giving $T_{AD}=1781.4$ N. Substitute T_{AD} into equation (4) gives $T_{AC}=1400.2$ N. Values of O_x, O_y and O_z are obtained by substituting the values of T_{AC} and T_{AD} into equations (1), (2) and (3).

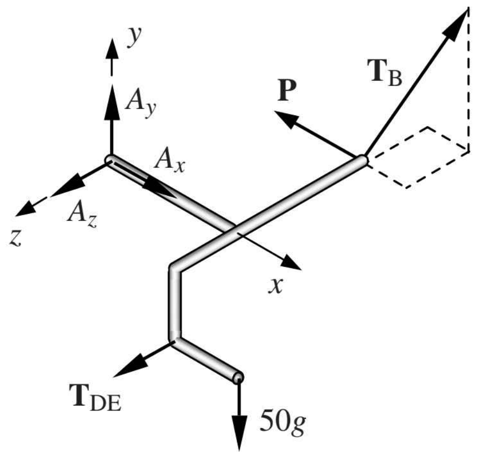
■ Example 10.2

The rigid body shown is in equilibrium. Determine the required mass m, force P, tension in cable DE and reaction at ball and socket joint at A.



Solution:

1. FBD



2. Solving components of the forces

Force P applied at point B, $\mathbf{P}=-P\text{ N }\mathbf{i}$

Force applied at point F, $-50g$ N $\mathbf{j}$

Force applied in cable DE, $\mathbf{T}_{\text{DE}} = T_{\text{DE}}$ N $\mathbf{k}$

Tension in the cable at point B, $\mathbf{T}_{\text{B}}$;

$$\begin{aligned} d_x &= 0.2 \text{ m} \\ d_y &= 0.6 \text{ m} \\ d_z &= -0.3 \text{ m} \\ \therefore d &= \sqrt{d_x^2 + d_y^2 + d_z^2} \\ &= \sqrt{(0.2)^2 + (0.6)^2 + (-0.3)^2} \\ &= 0.7 \text{ m} \end{aligned}$$

$$\begin{aligned} T_{\text{Bx}} &= T_{\text{B}}\frac{d_x}{d} = T_{\text{B}}\frac{(0.2)}{(0.7)} \\ T_{\text{By}} &= T_{\text{B}}\frac{d_y}{d} = T_{\text{B}}\frac{(0.6)}{(0.7)} \\ T_{\text{Bz}} &= T_{\text{B}}\frac{d_z}{d} = T_{\text{B}}\frac{(-0.3)}{(0.7)} \\ \therefore \mathbf{T}_{\text{B}} &= \left(\frac{2}{7}T_{\text{B}}\mathbf{i} + \frac{6}{7}T_{\text{B}}\mathbf{j} - \frac{3}{7}T_{\text{B}}\mathbf{k}\right) \text{ N} \end{aligned}$$

Reaction at point A; $\mathbf{A} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})$ N

3. Solving components of the moments

Taking moment about A gives;

$$\sum M_{\text{A}} = (\mathbf{r}_{\text{AB}} \times \mathbf{P}) + (\mathbf{r}_{\text{AD}} \times \mathbf{T}_{\text{DE}}) + (\mathbf{r}_{\text{AF}} \times 50\mathbf{g}) + (\mathbf{r}_{\text{AB}}) \times \mathbf{T}_B$$

moment about A due to force $\mathbf{P}$

$$r_{\text{AB}} \times \mathbf{P} = (0.6\mathbf{i} - 0.6\mathbf{k} \times (-P\mathbf{i})) = 0.6P\mathbf{j}$$

moment about A due to force $\mathbf{T}_{\text{DE}}$

$$r_{\text{AD}} \times \mathbf{T}_{\text{DE}} = (0.6\mathbf{i} - 0.3\mathbf{j} + 0.3\mathbf{k}) \times (T_{\text{DE}}\mathbf{k})) = -0.3T_{\text{DE}}\mathbf{i} - 0.6T_{\text{DE}}\mathbf{j}$$

moment about A due to force $50g$

$$r_{\text{AF}} \times -50g\mathbf{j} = (0.9\mathbf{i} - 0.3\mathbf{j} + 0.3\mathbf{k}) \times (-50g\mathbf{j}) = 147.15\mathbf{i} - 441.45\mathbf{k}$$

moment about A due to force $\mathbf{T}_{\text{B}}$

$$r_{\text{AB}} \times \mathbf{T}_{\text{B}} = (0.6\mathbf{i} - 0.6\mathbf{k}) \times \left(\frac{2}{7}T_{\text{B}}\mathbf{i} + \frac{6}{7}T_{\text{B}}\mathbf{j} - \frac{3}{7}T_{\text{B}}\mathbf{k}\right) = 0.514T_{\text{AB}}\mathbf{i} + 0.086T_{\text{AB}}\mathbf{j} + 0.514T_{\text{AB}}\mathbf{k}$$

For equilibrium $\sum M = 0$;

k component $0.514T_B - 441.45 = 0$
$T_B = 859$ N
$m = 87.6$ kg
i component $-0.3T_{DE} + 147.15 + 0.514T_B = 0$
$T_{DE} = 1962$ N
j component $0.6P - 0.6T_{DE} + 0.086T_B = 0$
$P = 1839$ N

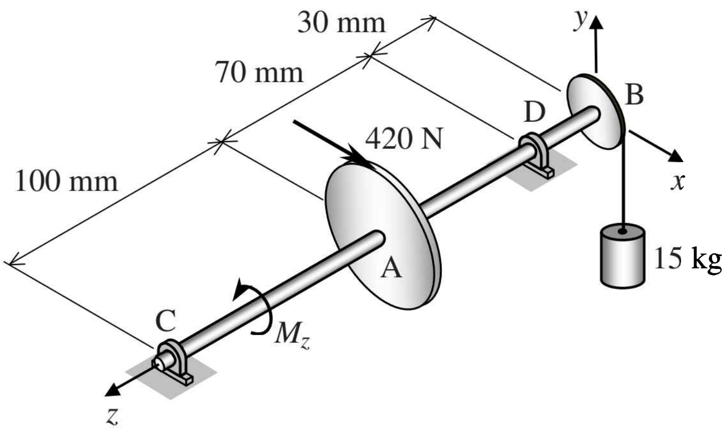
For equilibrium $\sum F = 0$;

i component $A_x - P + \frac{2}{7}T_B = 0$
$A_x = 1593.6$ N
j component $A_y - 50g + \frac{6}{7}T_B = 0$
$A_y = -246$ N
k component $A_z + T_{DE} - \frac{3}{7}T_B = 0$
$A_z = -1593.9$ N

Therefore $\mathbf{A} = 1593.6$ N**i** $-$ 246 N**j** $-$ 1593.9 N**k**

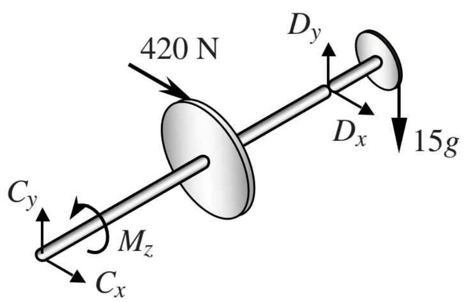
Example 10.3

Gear A of 50 mm radius and pulley B of 30 mm radius are attached to a shaft supported by bearings C and D as shown. Both bearings at C and D do not support any moment and axial load. Determine the required applied couple M_z and the reaction at bearings C and D for the system to maintain equilibrium.



Solution:

Draw the FBD of the problem:



Based on this FBD, the forces acting on the rigid body are:

- $-15g\mathbf{j}$
- $420\mathbf{i}$
- $\mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j}$
- $\mathbf{D} = D_x\mathbf{i} + D_y\mathbf{j}$

For equilibrium $\sum F = 0$;

$$\begin{array}{lll} \mathbf{i}\text{ component} & 420 + C_x + D_x = 0 & \ldots(1) \\ \mathbf{j}\text{ component} & C_y + D_y - 15g = 0 & \ldots(2) \end{array}$$

For equilibrium $\sum M_C = 0$;

$$\begin{aligned} \sum M_C =& (\mathbf{r}_{CA} \times 420\mathbf{i}) + (\mathbf{r}_{CB} \times -15g\mathbf{j}) + (\mathbf{r}_{CD} \times \mathbf{D}) + M_z\mathbf{k} \\ =& \{(0.05\mathbf{j} - 0.1\mathbf{k}) \times 420\mathbf{i}\} + \{(0.03\mathbf{i} - 0.2\mathbf{k}) \times -15g\mathbf{j}\} \\ &+ \{(-0.17\mathbf{k}) \times (D_x\mathbf{i} + D_y\mathbf{j})\} + M_z\mathbf{k} \\ =& (-42\mathbf{j} - 21\mathbf{k}) + (-29.43\mathbf{i} - 4.4145\mathbf{k}) + (0.1D_y\mathbf{i} - 0.1D_x\mathbf{j}) + M_z\mathbf{k} \end{aligned}$$

$$\begin{array}{ll} \mathbf{i}\text{ component} & -29.43 + 0.17D_y = 0 \\ & D_y = 173.12\text{ N} \\ \mathbf{j}\text{ component} & -42 - 0.17D_x = 0 \\ & D_x = -247.06\text{ N} \\ \mathbf{k}\text{ component} & -21 - 4.4145 + M_z = 0 \\ & M_z = 25.4145\text{ Nm} \end{array}$$

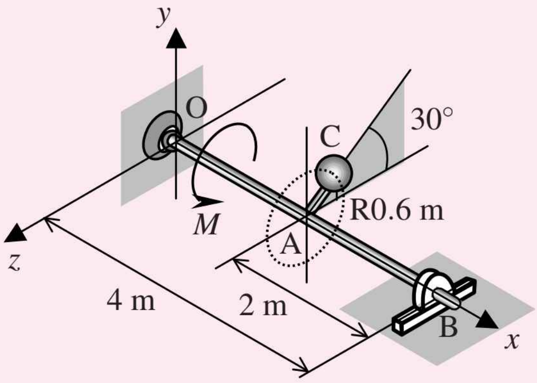
Substitute into (1) and (2) gives $C_x = -173$ N and $C_y = -26$ N. Hence;

$$\begin{aligned} M_z =& 25.4145\text{ Nm} \\ \mathbf{C} =& -173\text{ N}\mathbf{i} - 26\text{ N}\mathbf{j} \\ \mathbf{D} =& -247.06\text{ N}\mathbf{i} + 173.12\text{ N}\mathbf{j} \end{aligned}$$

10.3 Example questions

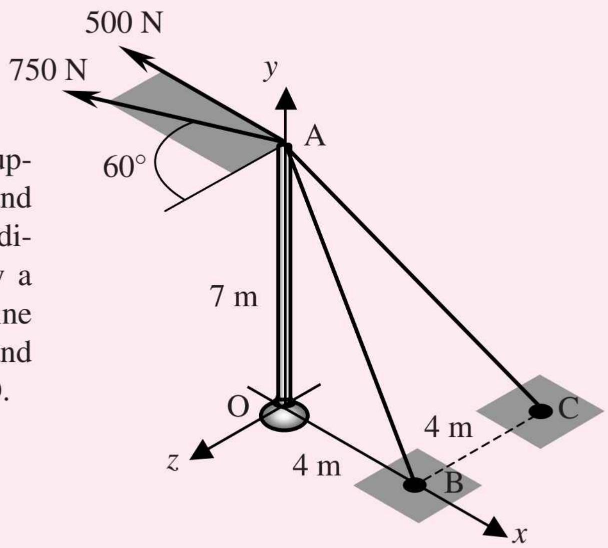
Exercise 10.1

The figure shows a uniform rod OAB with a mass of 5 kg/m supported by a ball–and–socket joint at O and a radial bearing at B, and loaded with moment $M = 100$ Nm. A light rod AC with a 5 kg mass at C is welded at A. Determine the reactions at the ball and socket joint at O and the bearing at B.



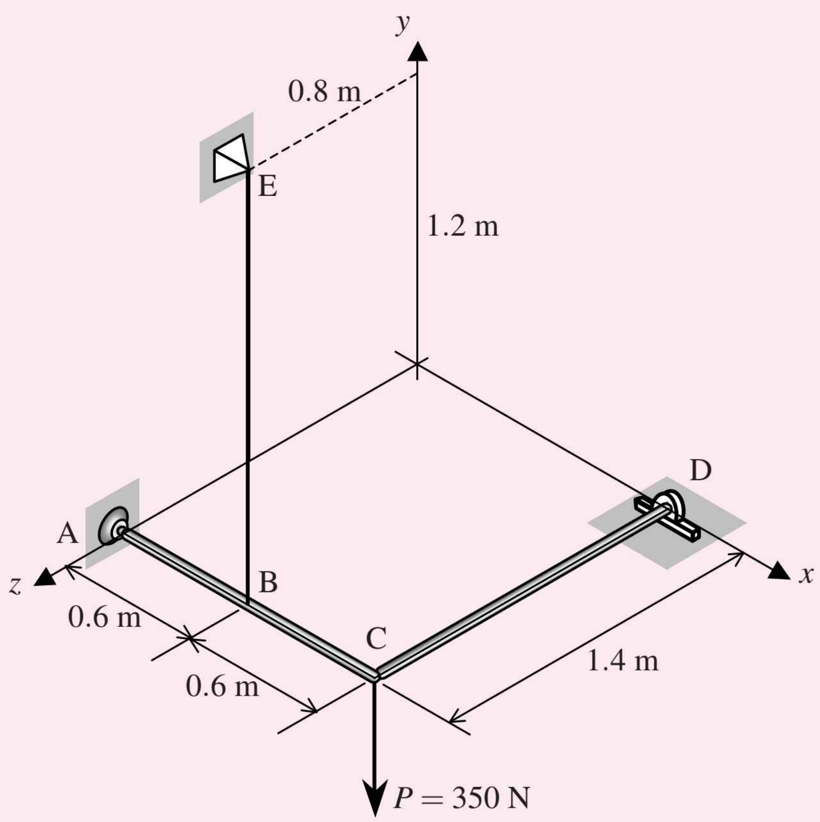
Exercise 10.2

Rod OA of negligible mass is supported by cables AB and AC and loaded with forces as shown in the diagram. If the rod is supported by a ball–and–socket joint at O, determine the tension in cables AB and AC, and the components of the reaction at O.



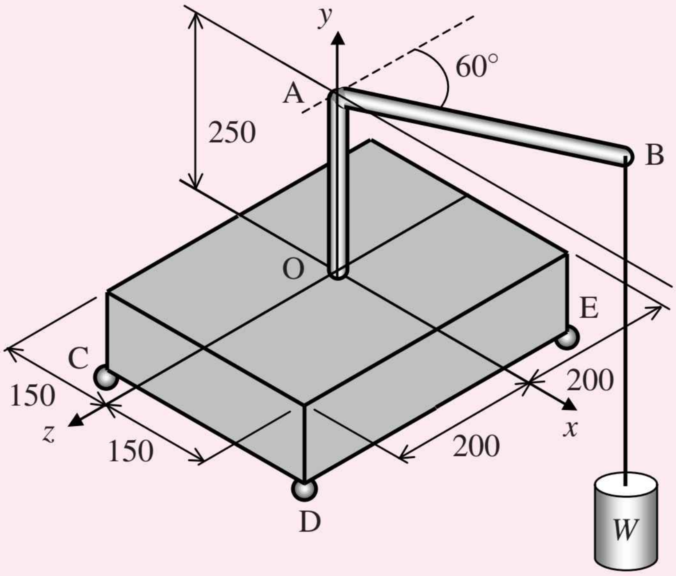
Exercise 10.3

The bent rod ABCD is supported by a ball–and–socket joint at A , cable BE at B, a radial bearing at D and loaded with force P at C. The bearing at D does not support any axial thrust or moment. Determine the reaction at A and D, and the tension in cable BE if the system shown is in equilibrium.



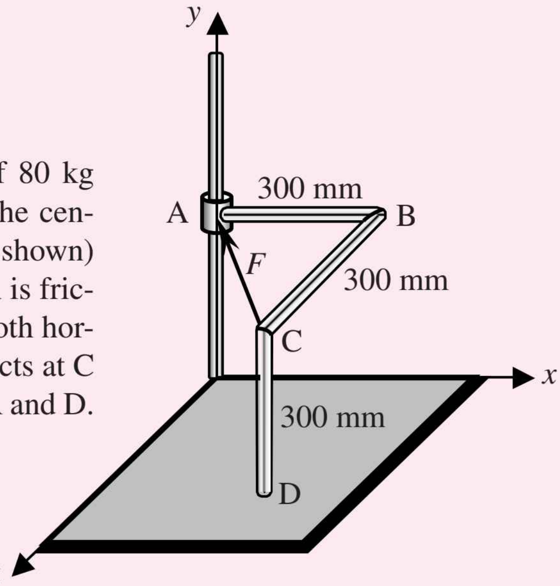
Exercise 10.4

A 4-wheel mechanism (one is hidden from view) is supporting a weight W at the end of arm AB. The maximum weight that can be supported without the mechanism tilting is $W = 20$ kg. Determine the length AB and the reactions at D and E. The weight of the whole mechanism is 400 N acting at O. All dimensions in mm.
Answer: $l_{AB} = 526$ mm, $F_D = 169$ N and $F_E = 427$ N



Exercise 10.5

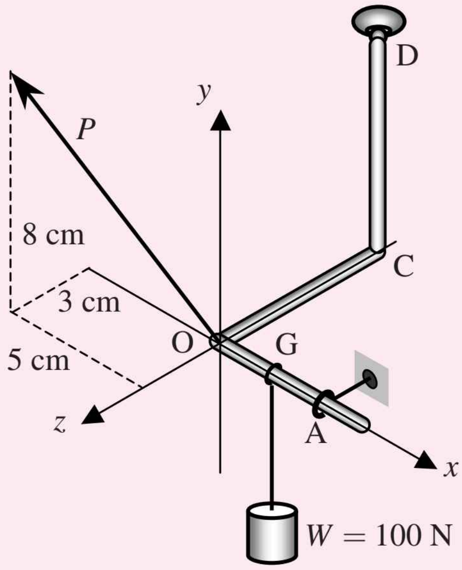
A uniform rod ABCD with a mass of 80 kg is shaped as shown in the diagram. The centre of gravity is located at point G (not shown) (250 mm, 250 mm, 150 mm). Collar A is frictionless and end D is located on a smooth horizontal surface. The force $F = 600$ N acts at C along CA. Determine the reactions at A and D.



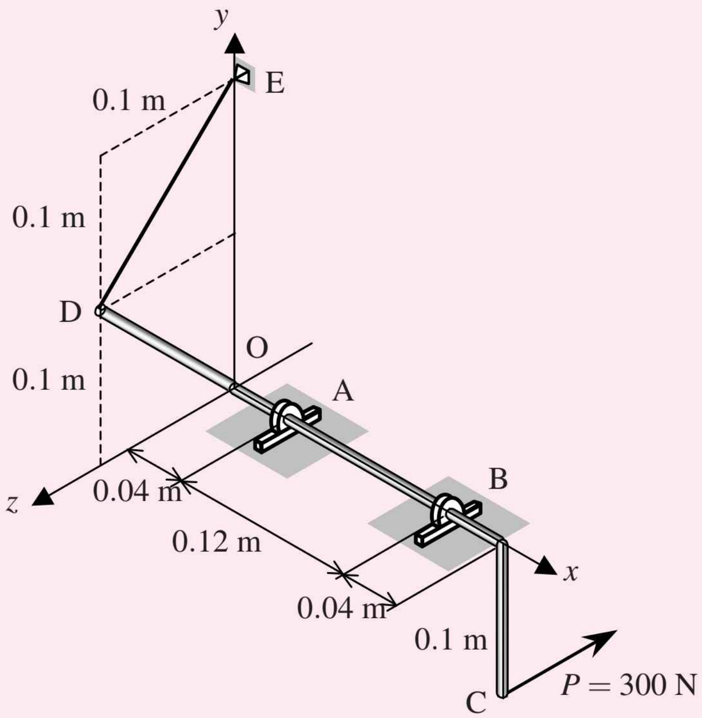
Exercise 10.6

A bent rod AOCD with a ball–and–socket joint at D, a ring at A and a force P, are used to support a load $W = 100$ N hanging at G. Determine the magnitude of force P by taking moment about the line AD.

Given:
AG = GO = 4 cm
OC = 12 cm
CD = 14 cm

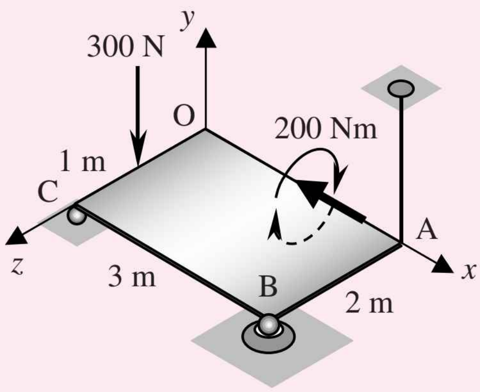


Exercise 10.7
The bent rod in the diagram is supported by two radial bearings at A and B, and cable DE. The system is loaded with a force $P = 300$ N at C which is parallel to the z-axis. Both bearings do not support any moment. Determine the reactions at A and B, and the tension in cable DE.



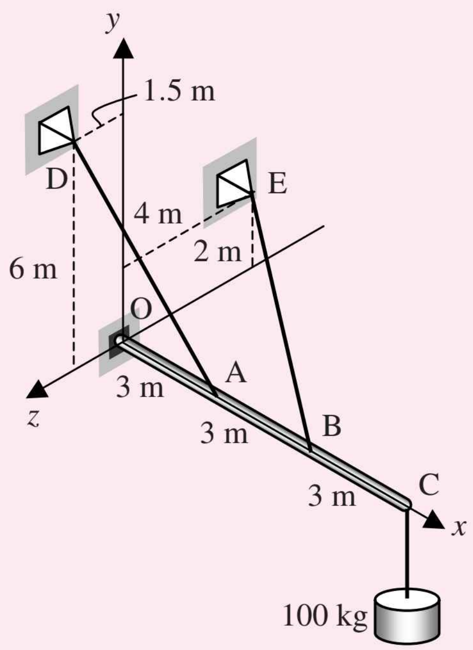
Exercise 10.8

A homogeneous metal plate of mass 100 kg is supported by a cable at A, a smooth roller at C and a ball–and–socket joint at B. If a 300 N force and a 200 Nm couple are exerted on the plate, determine the tension of the cable at A and the reactions at B and C.



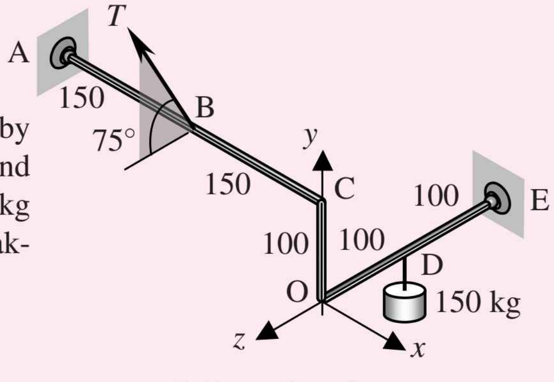
Exercise 10.9

Rod OABC with a fixed support at O is loaded with a 100 kg mass at C and supported by cables AD and BE. The tension in both cables are 5000 N each. Determine the reaction at O due to all the forces acting on the rod.



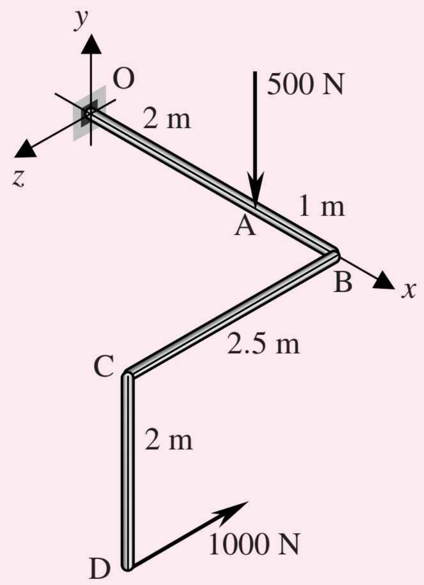
Exercise 10.10

The bent rod ACOE is supported by ball–and–socket joints at A and E, and force T at B. Rod is loaded with a 150 kg mass at D. Determine the force T by taking moment about the AE axis.



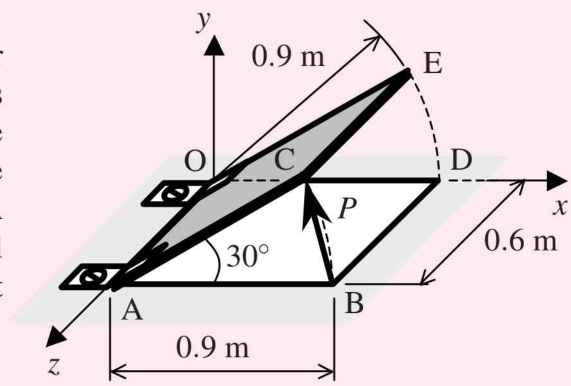
Exercise 10.11

Two loads are exerted on the bent rod OBCD, 500 N at A and 1000 N at D. The bent rod is fixed supported at O. Determine the components of the reaction at O. Answer: $O_x = 0$, $O_y = 500$ N, $O_z = 1000$ N, $M_x = -2000$ Nm, $M_y = -3000$ Nm and $M_z = 1000$ Nm.



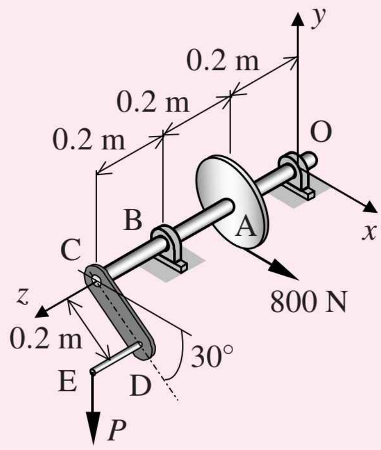
Exercise 10.12

The 20 kg rectangular manhole cover OACE supported by hinges O and A is maintained at the 30° position by the force P at B. Determine the magnitude of force P and the reactions at O and A if bearing A does not support any axial load and both bearings do not support any moment.



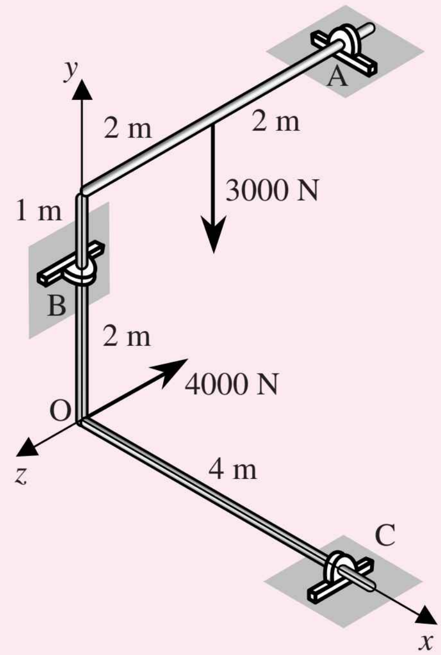
Exercise 10.13

The winch shown is held in equilibrium by a vertical force P applied at point E. Determine the magnitude of force P and the reaction at radial bearings O and B. Both bearings do not support axial thrust or couples. Given DE = 0.2 m and diameter of the disc is 0.2 m.



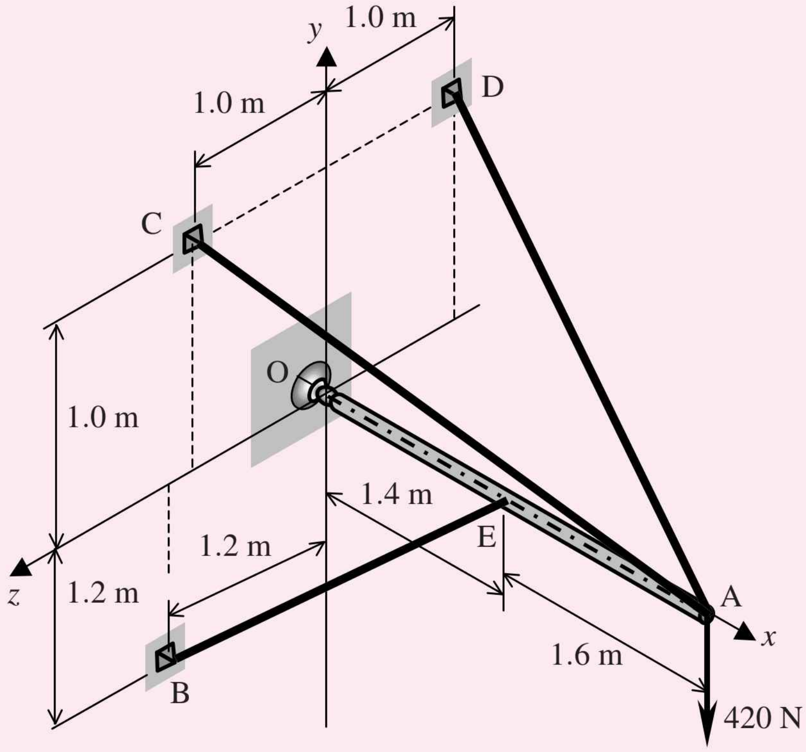
Exercise 10.14

A bent rod is supported by three radial bearings at A, B and C. All bearings do not support any axial thrust or moment. If the rod is loaded with forces as shown, determine the reactions on each bearing.



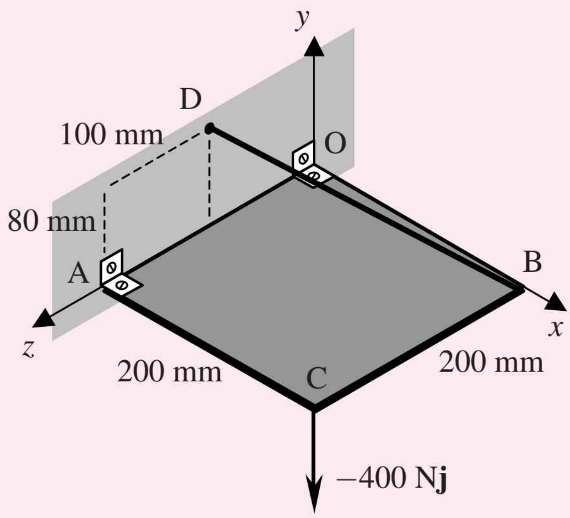
Exercise 10.15

Boom OA of negligible mass is supported by a ball–and–socket joint at O and cables EB, AC and AD as shown in the figure. The ratio of the tension in cables AD and AC is 3:1. Calculate the tension in cables EB, AC and AD, and the reaction at the ball–and–socket joint at O if a load of 420 N is applied at end A.



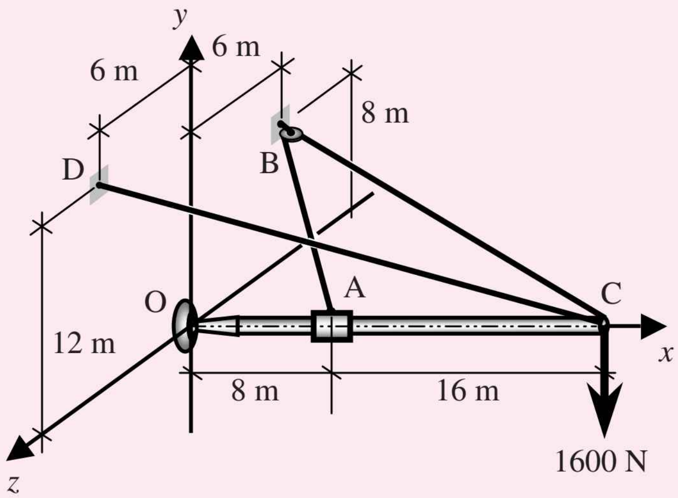
Exercise 10.16

A uniform plate OACB is supported by two hinges at O and A and by cable BD. Both hinges do not support any moment and the hinge at A does not exert any axial thrust. Determine the reactions at both hinges and the tension in cable BD.



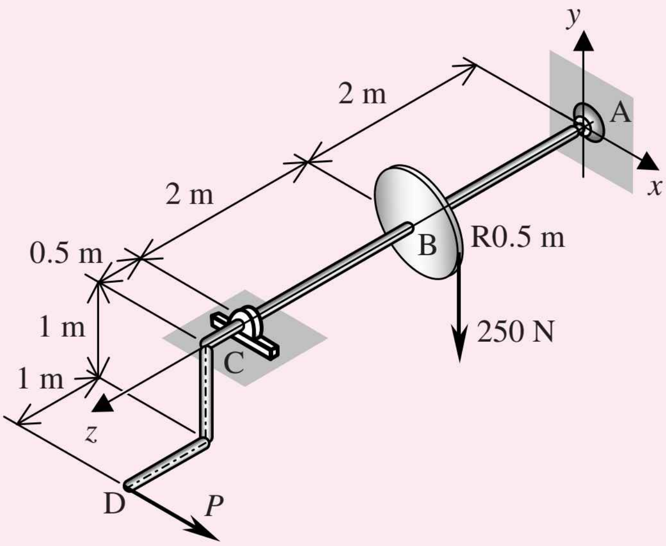
Exercise 10.17

Rod OC is supported by a ball and–socket–joint at O and cables ABC and CD. The cable ABC passed through a pulley at B as shown in the diagram. Determine the tension in the cables and components of the reaction at O if a 1600 N force is applied at C.



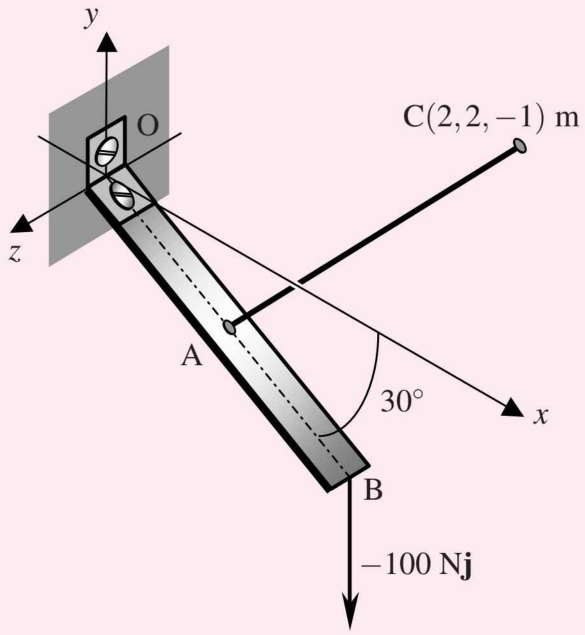
Exercise 10.18

The bent rod shown is supported by a ball–and–socket joint at A, a radial bearing at C and acted upon by a force P at end D and a 250 N force at pulley B. Determine the reactions at A and C, and the force P if the system is in equilibrium.



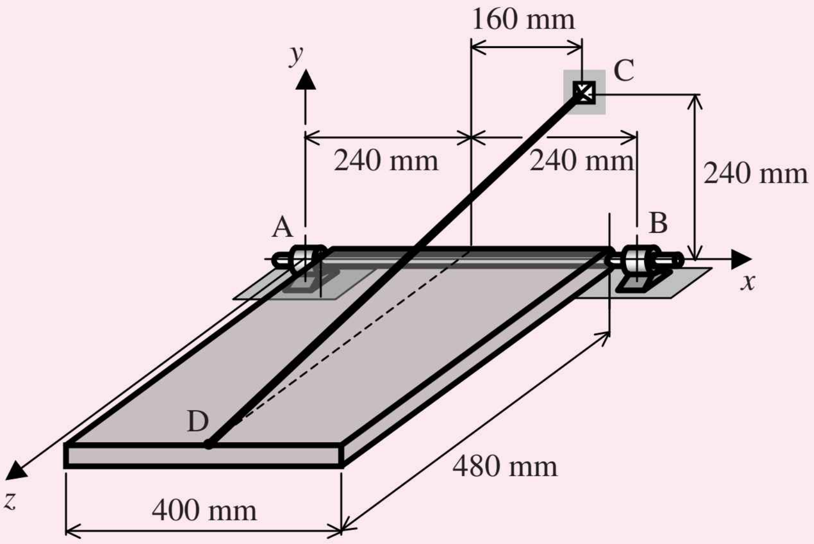
Exercise 10.19

A light bar OB of 400 cm length is supported by a hinge at O and cable AC at A. The hinge lies on the z-axis. The centre line of the bar lies on the *x*-*y* plane and A is the mid point of OB. The hinge support both moments and axial thrust. Determine the tension in cable AC and components of the reaction at O.



Exercise 10.20

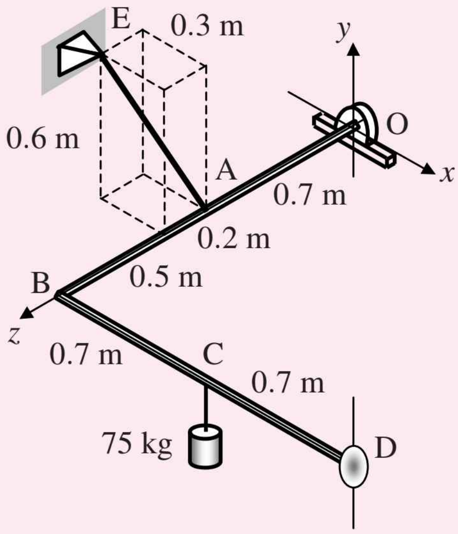
The 30 kg homogeneous plate with uniform thickness is welded to the shaft AB as shown. The plate is maintained at the horizontal position by cable DC. Bearing A does not support any axial load and both bearings do not support any moment. Determine the tension in cable CD and the reactions at bearings A and B.



Exercise 10.21

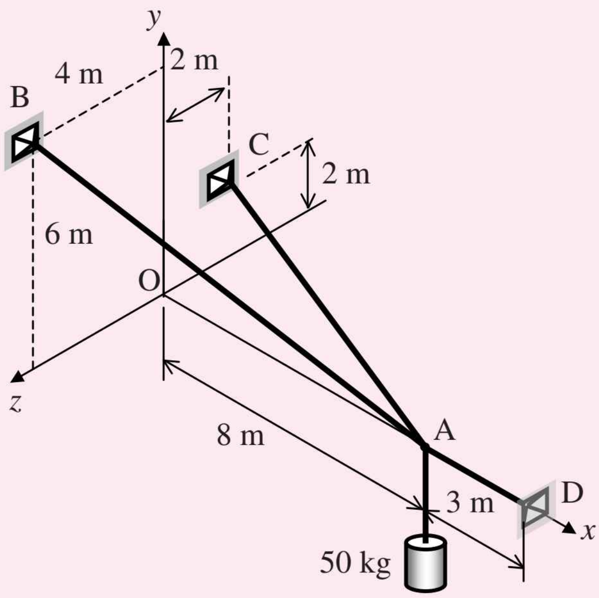
The bent rod shown in the figure is supported by a radial bearing at O, a ball–and–socket joint at D and by cable AE at A. A 75 kg mass acts at C. Determine

a. the tension in cable AE.
b. the reaction at O and D.



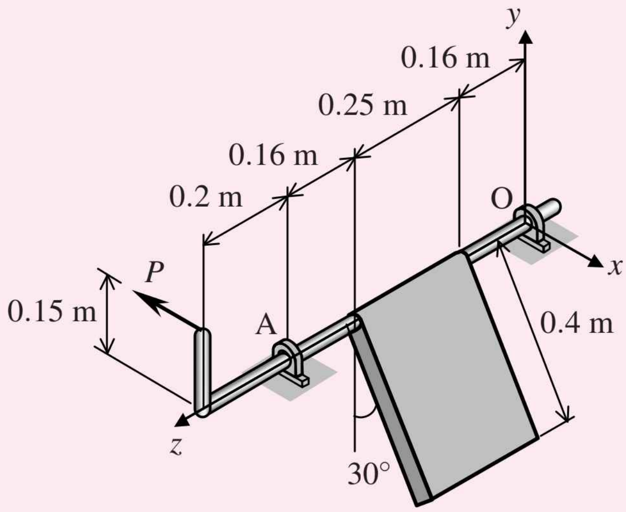
Exercise 10.22

Determine the tension in cables AB, AC and AD required to hold the 50 kg mass in equilibrium.



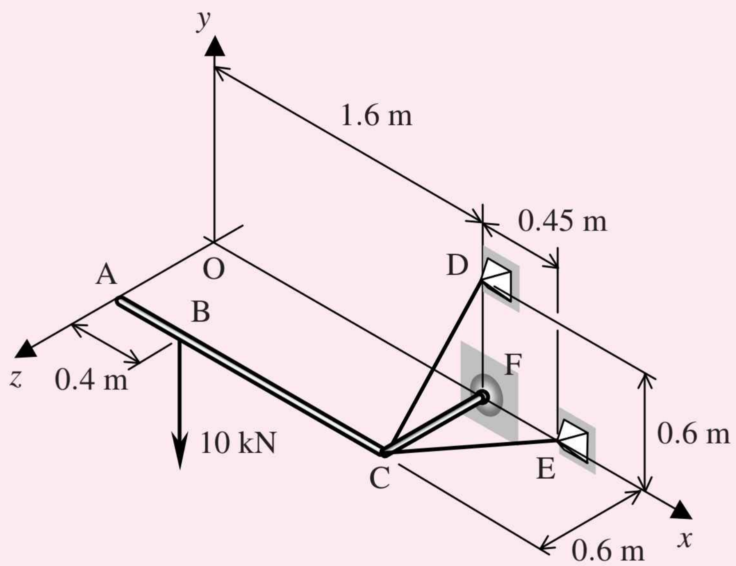
Exercise 10.23

A 12 kg, 0.25 m × 0.4 m homogenous plate is welded on to a bent rod and supported by two bearings A and O as shown. Bearing A does not support any axial thrust. Determine the force P to maintain equilibrium.



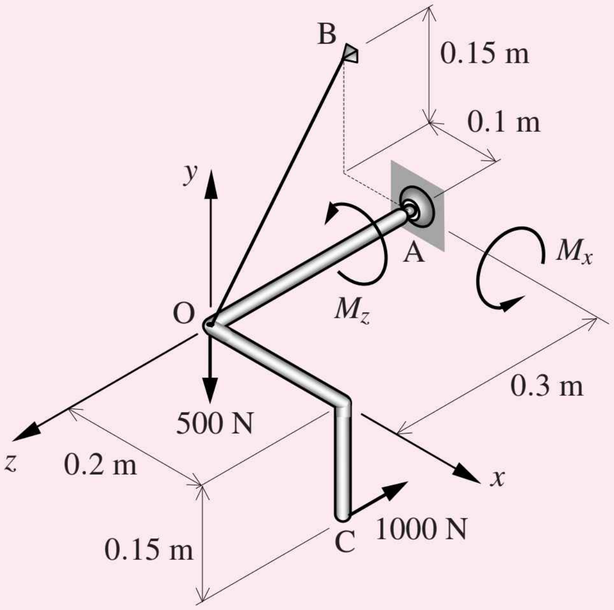
Exercise 10.24

The figure shows a bent rod supported by a ball–and–socket joint at F and cables; CD and CE, while loaded with a 10 kN force at B. Determine the reaction at the ball–and–socket joint and the tension in all cables.



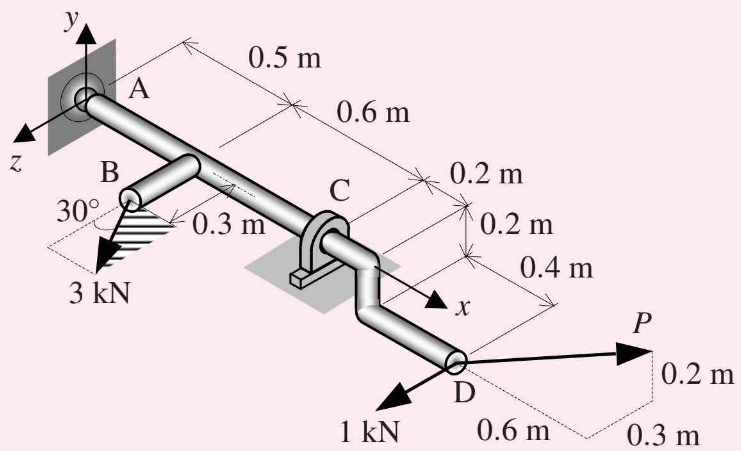
Exercise 10.25

The system shown is in equilibrium. Determine the reaction at ball and socket joint A, the tension in cable OB and the moments M_x and M_z to support the 500 N and 1000 N forces at O and C respectively.



Exercise 10.26

The bent rod shown is supported by a ball-and-socket joint at A and a radial bearing at C, and loaded with forces 3 kN at B and, 1 kN and P at D. The bearing at C does not support any axial thrust or moment. Determine the reaction at A and C, and the force P if the system shown is in equilibrium.



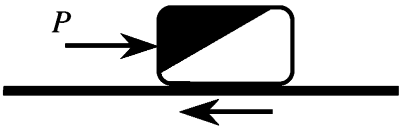
11 — Friction force

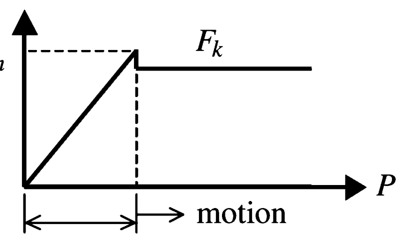
11.1 Introduction on friction

Friction **only exist** when solid bodies/surfaces/fluids are **in contact with** or **touching** each other. There are two types of frictions which are

- **Dry friction** - friction between solid bodies/surfaces
- **Fluid friction** - friction in fluids

Due to the friction, a frictional force exist that acts against the motion of an object.





Consider the figure above where an external force P is applied to an object placed on a rough surface with a coefficient of friction μ. Force P tend to move the object to the right but the object is still in equilibrium state (not moving) as force P is being opposed by the friction force F due to the surface roughness. Magnitude of friction force F is the same as force P but acting in opposite direction (against the direction of motion). If magnitude of force P is increased to a certain value, the magnitude of friction force F also increased to the same value. However there is a maximum limit for the value of the friction force which is given by

$$F_m = \mu_s N$$

where F_m is the maximum static friction force, μ_s is the static coefficient of friction and N is the normal force acting on the contacting surfaces. Once the magnitude of force P is larger than F_m, the object starts moving and the coefficient of friction becomes μ_k

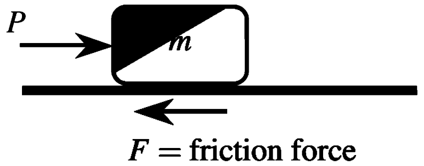
which is the kinetic coefficient of friction. Generally $\mu_k < \mu_s$, hence once the object starts moving, the magnitude of friction force resisting the motion becomes less and constant given by

$$F_k = \mu_k N$$

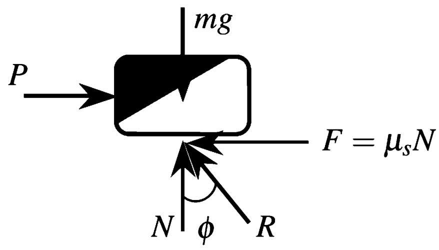
General solution procedure for friction problem

- Draw the FBD.
- Make sure that the friction force is directed against direction of motion.
- For cases involving two bodies or more, draw the FBD similar to answering frame/machine problems.

Physical problem



Free Body Diagram (FBD)



A free body diagram of the forces acting on the body is shown in the figure on the right. Friction angle (ϕ) is the angle between the normal force, N and the resultant, R (between N and F).

$$\phi = \tan^{-1}\frac{\mu_s N}{N} = \tan^{-1}\mu_s$$

which give the static coefficient of friction as;

$$\mu_s = \tan\phi$$

Experiments conducted verify that the maximum static friction force (F_m) is directly proportional to the normal force as given by $F_m = \mu_s N$. Both coefficients of friction μ_s and μ_k are independent of surface area of contact but are exceptionally dependent of the roughness of the surface of contact. Therefore, the values used are assumptions based on practical experimental investigations.

Table 11.1 shows the values of coefficient of static friction μ_s for various contacting surfaces. The corresponding coefficient of kinetic friction μ_k is approximately 25% lower than the given values of coefficient for static friction μ_s.

Contacting surfaces	**Coefficient of static friction μ_s**
metal on metal	0.15 – 0.60
metal on wood	0.20 – 0.60
metal on rock	0.30 – 0.70
metal on leather	0.30 – 0.60
wood on wood	0.25 – 0.50
wood on leather	0.25 – 0.50
rock on rock	0.40 – 0.70
earth on earth	0.20 – 1.00
rubber on concrete	0.60 – 0.90

Table 11.1: Approximate values of coefficient of static friction μ_s for various contacting surfaces.

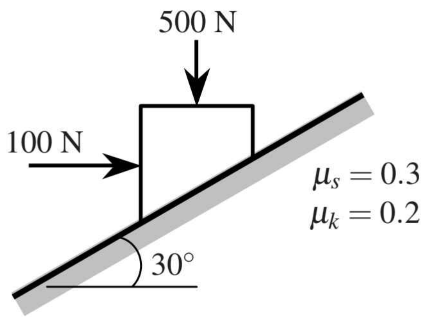
11.2 Categories of problem involving friction

11.2.1 Category 1

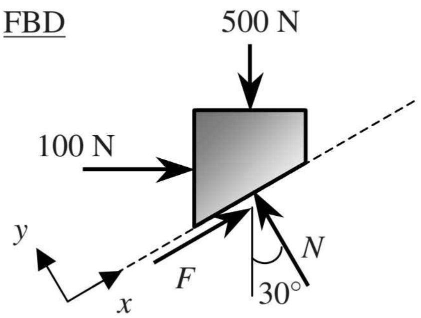
All external forces, and both coefficients of friction (μ_s and μ_k) are known. Determine whether the system is in equilibrium or in motion.

■ Example 11.1

Determine whether the block is in equilibrium or in motion. Hence, find the magnitude and direction of the friction force.



Solution:



Important observations:

- For category 1, the friction force is represented by F, not $\mu_s N$.
- In this case, F is assumed up the incline meaning, the block is assumed to move or tend to move down the incline.

$$(+\nearrow)\sum F_x = 0$$
$$100\cos 30^\circ + F - 500\sin 30^\circ = 0$$
$$\therefore F = 163.4 \text{ N}$$

Due to the assumed direction for friction force F, there are two possibilities for it's magnitude:

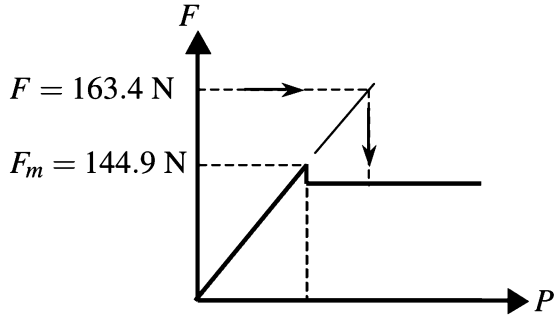
- (+ve) answer → the assumed direction of F is correct.
- (–ve) answer → the assumed direction of F is incorrect, reverse direction of F. In this question, F is directed up the incline and the block is moving or tend to move down the incline.

$$(+\nwarrow)\sum F_y = 0$$
$$N - 100\sin 30^\circ - 500\cos 30^\circ = 0$$
$$\therefore N = 483 \text{ N}$$

Compare the magnitude of F with the magnitude of maximum static friction force F_m. There are two possibilities:

- $F > F_m \rightarrow$ the block is moving down the incline, hence $F = F_k = \mu_k N$.
- $F_m > F \rightarrow$ the block tends to move down the incline (still in equilibrium), hence $F =$ the calculated F.

for this particular example;



The maximum static friction force is given by

$$F_m = \mu_s N = (0.3)(483) = 144.9 \text{ N}$$

Since 163.4 N > 144.9 N, therefore $F > F_m$ whereby the block is moving down the incline. Hence the actual friction force is

$$F = F_k = \mu_k N = (0.2)(483) = 96.6 \text{ N}\angle 30^\circ$$

Additional notes:

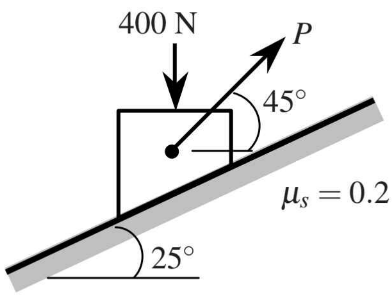
- Choose a suitable pair of axes.
- Reverse the direction of F (and motion) if the answer is negative.

11.2.2 Category 2

The system is on the verge of moving (the system is in equilibrium) with the coefficient of static friction (μ_s) known. Determine the required force to maintain it.

▪ Example 11.2

Determine the range of P so that the system is in equilibrium.



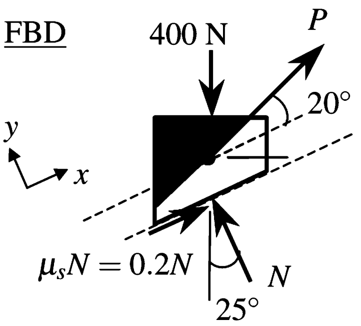
Solution:
There are two possibilities:

a. The block is on the verge of moving down the incline. Friction force, $F = F_m = \mu_s N$

directed up the incline.

b. The block is on the verge of moving up the incline. Friction force, $F = F_m = \mu_s N$ directed down the incline.

Case (a)

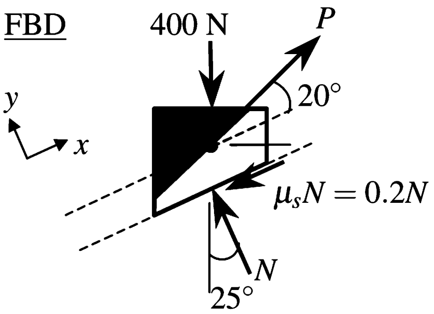


$$(+\nearrow)\sum F_x = 0$$
$$0.2N - 400\sin 25^\circ + P\cos 20^\circ = 0$$
$$0.2N - 169 + 0.94P = 0 \ldots (1)$$

$$(+\nwarrow)\sum F_y = 0$$
$$N - 400\cos 25^\circ + P\sin 20^\circ = 0$$
$$N - 362.5 + 0.34P = 0 \ldots (2)$$

Solving both simultaneous equations (1) and (2) gives $P = 110.7$ N.

Case (b)



$$(+\nearrow)\sum F_x = 0$$
$$-0.2N - 400\sin 25^\circ + P\cos 20^\circ = 0$$
$$-0.2N - 169 + 0.94P = 0 \ldots (3)$$

$$(+\nwarrow)\sum F_y = 0$$
$$N - 400\cos 25^\circ + P\sin 20^\circ = 0$$
$$N - 362.5 + 0.34P = 0 \ldots (4)$$

Solving both simultaneous equations (3) and (4) gives $P = 239.6$ N.

Therefore the range of force P is $110.7 \text{ N} \leqslant P \leqslant 239.6 \text{ N}$

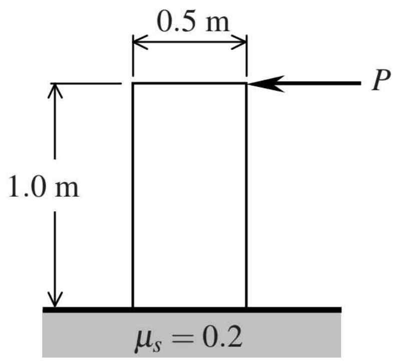
11.2.3 Category 3

In this category, the object has a possibility either to tilt or slide. Hence, this will be determined by using the moment equation. Observe that dimensions of the object is given in this type of problem.

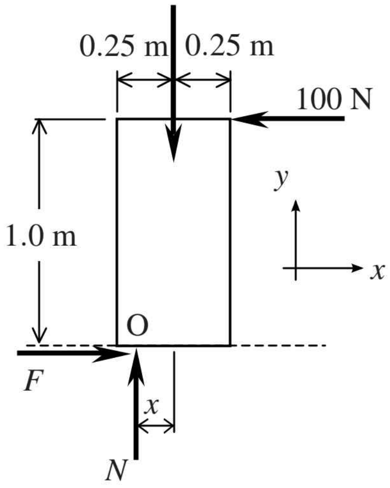
■ Example 11.3

Force P is acting on a uniform 80 kg block as shown. Determine whether the block remains in equilibrium when

(a) $P = 100$ N
(b) $P = 175$ N
(c) $P = 200$ N



Solution:
(a) for $P = 100$ N
FBD



Notes:

- The friction force, $F \neq \mu_s N$.
- Notice that there are 3 unknowns.
- Direction of the normal force N is not the center of the block.
- Solve the three equilibrium equations first.

$(+\rightarrow)\sum F_x = 0$
$F - 100 = 0$
$\therefore F = 100\ \text{N}(\rightarrow)$

$(+\uparrow)\sum F_y = 0$
$N - 784.8 = 0$
$\therefore N = 784.8\ \text{N}(\uparrow)$

$(+\circlearrowleft)\sum M_O = 0$
$784.8(x) - 100(1) = 0$
$\therefore x = 0.1274$ m

Then the value of x is compared with $\frac{1}{2}$width:

If $x \geqslant \frac{1}{2}$ width $\rightarrow$ Tilting will occur. Else if $x < \frac{1}{2}$width $\rightarrow$ Check for slipping.

If $F > F_m \rightarrow$ Slipping occurs.
If $F < F_m \rightarrow$ Not moving/in equilibrium.

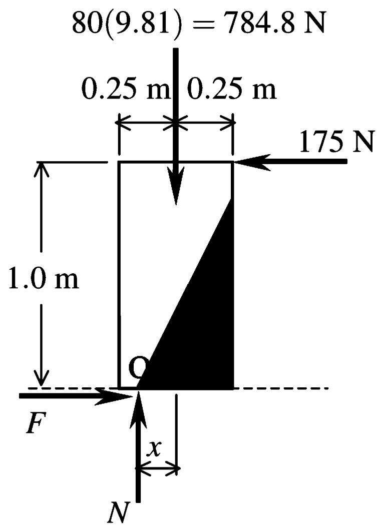
For this particular example, the result is $x = 0.1274$ m. Comparing with half of the block width (0.25 m), the value of x is smaller ($x \leqslant 0.25$ m). Therefore the block does not tilt. The maximum static friction force is given by

$$F_m = \mu_s N = 0.2(784.4) = 156.96 \text{ N}$$

Since $F = 100$ N and $F_m > F$, the block is still in equilibrium.

(b) for $P = 175$ N

FBD



$(+\rightarrow)\sum F_x = 0$
$F - 175 = 0$
$\therefore F = 175 \text{ N}(\rightarrow)$

$(+\uparrow)\sum F_y = 0$
$N - 784.8 = 0$
$\therefore N = 784.8 \text{ N}(\uparrow)$

$(+\circlearrowright)\sum M_O = 0$
$784.8(x) - 175(1) = 0$
$\therefore x = 0.223 \text{ m}$

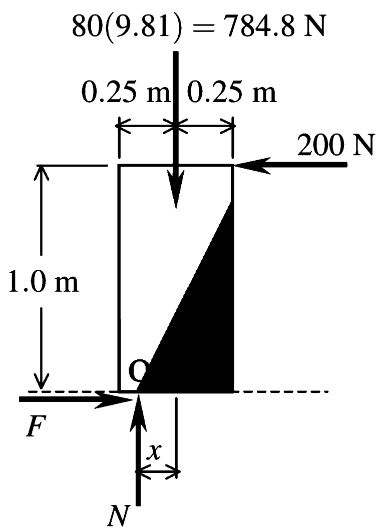
For this particular example, the result is $x = 0.223$ m. Comparing with half of the block width (0.25 m), the value of x is smaller ($x \leqslant 0.25$ m). Therefore the block does not tilt. The maximum static friction force is given by

$$F_m = \mu_s N = 0.2(784.4) = 156.96 \text{ N}$$

Since $F = 175$ N and $F > F_m$, the block is slipping (in motion).

(c) for $P = 200$ N

FBD



$(+\rightarrow)\sum F_x = 0$
$F - 200 = 0$
$\therefore F = 200 \text{ N}(\rightarrow)$

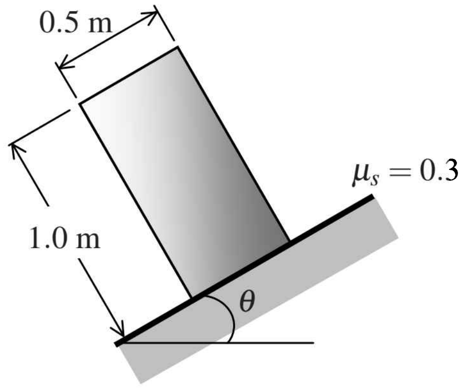
$(+\uparrow)\sum F_y = 0$
$N - 784.8 = 0$
$\therefore N = 784.8 \text{ N}(\uparrow)$

$(+\circlearrowright)\sum M_O = 0$
$784.8(x) - 200(1) = 0$
$\therefore x = 0.255 \text{ m}$

For this particular example, the result is $x = 0.255$ m. Comparing with half of the block width (0.25 m), the value of x is larger ($x > 0.25$ m). Therefore the block tilts. In other words, normal force N no longer exist when block tilts.

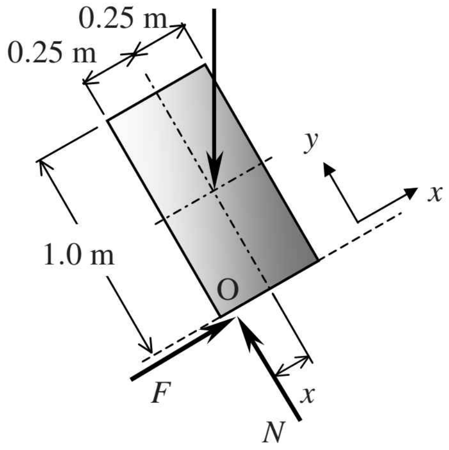
■ Example 11.4

The 80 kg block rests on an incline as shown. Determine the largest angle θ before the block moves.



Solution:

FBD



Notes:

- The friction force, $F \neq \mu_s N$.
- Notice that there are 4 unknowns.
- Direction of the normal force N is not to the center of the block.
- Apply the three equilibrium equations first.

$(+\nearrow)\sum F_x = 0$

$F - 784.8\sin\theta = 0$

$\therefore F = 784.8\sin\theta$ N($\nearrow$)

$(+\nwarrow)\sum F_y = 0$

$N - 784.8\cos\theta = 0$

$\therefore N = 784.8\cos\theta$ N($\nwarrow$)

$(+\circlearrowleft)\sum M_O = 0$

$784.8\cos\theta(x) - 784.8\sin\theta(0.5) = 0$

Assume the block slides

If the block slides,

$$F = F_m = \mu_s \text{N} = 0.3(784.8\cos\theta)$$

by solving all four equations yields

$$\theta = 16.7^\circ \qquad N = 751.7 \text{ N} \qquad F = 225.5 \text{ N} \qquad x = 0.15 \text{ m}$$

compare x with $\frac{1}{2}$width, since $x = 0.15$ m < 0.25 m, the block will slide and $\theta = 16.7^\circ$.

Assume the block tilts

If the block tilts, $x = 0.25$ m. Solving all four equations yields

$$x = 0.25 \text{ m} \qquad \theta = 26.6^\circ \qquad N = 701.7 \text{ N} \qquad F = 351.4 \text{ N}$$

comparing F with F_m;

$$F = 351.4 \text{ N} \qquad \text{and} \qquad F_m = \mu_s N = 0.3(701.7) = 210.5 \text{ N}$$

since $F > F_m$, the block will slide first and $\theta = 16.7^\circ$. In other words, the smaller θ is the answer. For problems where θ is known, there are 3 possibilities, i.e the block is in equilibrium, slides (in motion) or tilts.

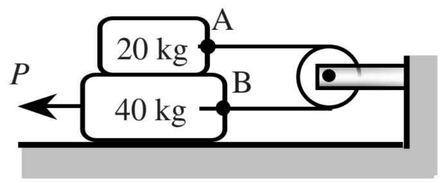
11.2.4 Category 4

In this category, friction involves two or more bodies. Free body diagram (FBD) for bodies involved are generally required.

■ **Example 11.5**

Determine the smallest force P to start moving the 40 kg mass,

a. with cable AB intact,
b. with cable AB removed.

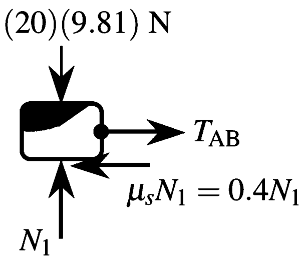


$\mu_s = 0.4$ and $\mu_k = 0.3$
(for all contacting surfaces)

Solution:

(a) Cable AB intact:

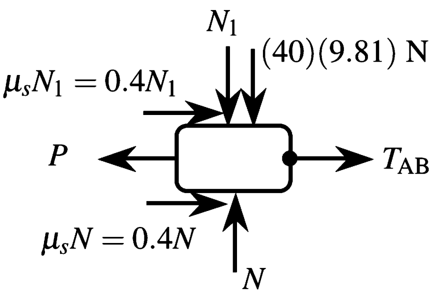
FBD for 20 kg mass:



Taking the summation of forces along x and y axes yields;

$$(+\uparrow)\sum F_y = 0$$
$$N_1 - (20)(9.81) = 0$$
$$N_1 = 196.2\ \text{N}$$
$$(+\rightarrow)\sum F_x = 0$$
$$T_{AB} - 0.4N_1 = 0$$
$$\therefore T_{AB} = 78.48\ \text{N}$$

FBD for 40 kg mass:

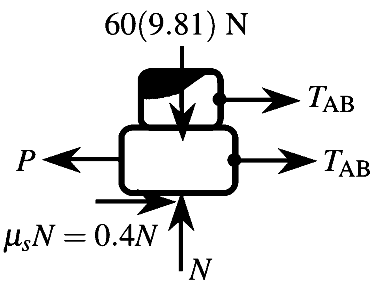


Taking the summation of forces along x and y axes yields;

$$(+\uparrow)\sum F_y = 0$$
$$-196.2 - (40)(9.81) + N = 0$$
$$N = 588.6\ \text{N}$$
$$(+\rightarrow)\sum F_x = 0$$
$$78.48 - P + 0.4(196.2) + 0.4N = 0$$
$$\therefore P = 392.4\ \text{N}$$

After solving the 20 kg mass FBD, it is also possible to solve the problem by drawing the overall FBD (both masses together).

Overall FBD



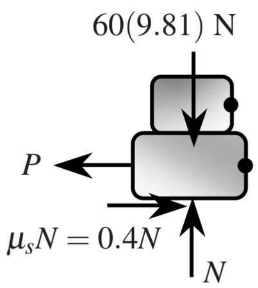
Taking the summation of forces along x and y axes yields;

$$(+\uparrow)\sum F_y = 0$$
$$N - (60)(9.81) = 0$$
$$N = 588.6\ \text{N}$$
$$(+\rightarrow)\sum F_x = 0$$
$$2(78.48) - P + 0.4N = 0$$
$$\therefore P = 392.4\ \text{N}$$

(b) Cable AB removed:

When cable AB is removed, only the overal FBD is needed.

Overall FBD



Taking the summation of forces along x and y axes yields;

$$(+\uparrow)\sum F_y = 0$$
$$N-(60)(9.81) = 0$$
$$N = 588.6\ \text{N}$$
$$(+\rightarrow)\sum F_x = 0$$
$$-P+0.4N = 0$$
$$\therefore P = 235.4\ \text{N}$$

Important notes

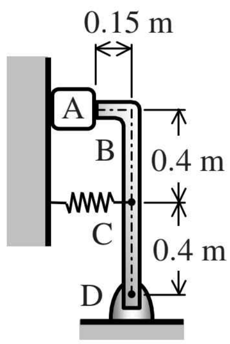
- Friction force is always directed against the direction of motion.
- $F = F_m = \mu_s N$ for all categories (except categories 1 and 3).
- Analysis for categories 1 and 4 are based on analysis of equilibrium of particle (without taking moments).
- The choices between 'common' or slanting axes depends on simplicity of determining force components.

11.2.5 Category 5

In this category, friction involves frame/machine problems.

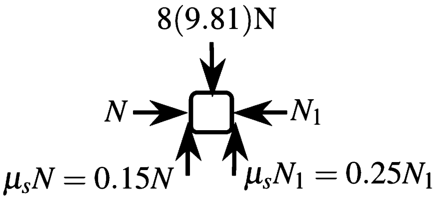
Example 11.6

Position of the 8 kg block A at the wall is maintained by member BCD. If the coefficients of static friction between block A and surface at B is $\mu_s = 0.25$, and between block A and the wall is $\mu_s = 0.15$, and block A is on the verge of falling, determine the force in the spring and the reactions at B and D for the instant shown.



Solution:

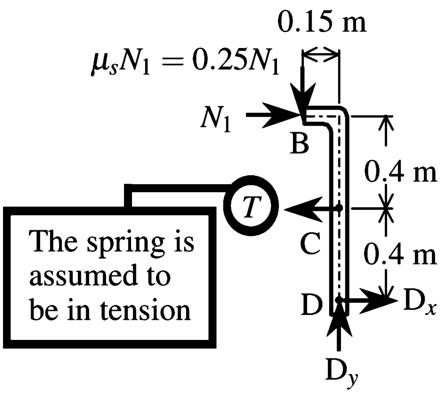
FBD block A



Taking the summation of forces along x and y axes yields;

$$(+\rightarrow)\sum F_x = 0$$
$$N - N_1 = 0$$
$$\therefore N = N_1$$
$$(+\uparrow)\sum F_y = 0$$
$$0.15N + 0.25N_1 - (8)(9.81) = 0$$
$$0.4N = (8)(9.81)$$
$$\therefore N = 196.2\text{ N} = N_1$$

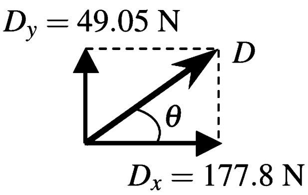
FBD member BCD



Taking the summation of forces along x and y axes yields;

$$(+\rightarrow)\sum F_x = 0$$
$$D_x - T + N_1 = 0$$
$$D_x - 374 + 196.2 = 0$$
$$\therefore D_x = 177.8\text{ N}(\rightarrow)$$
$$(+\uparrow)\sum F_y = 0$$
$$D_y - 0.25N_1 = 0$$
$$D_y - 0.25(196.2) = 0$$
$$\therefore D_y = 49.05\text{ N}(\uparrow)$$

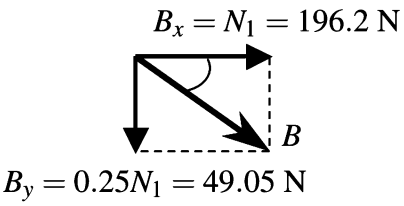
Reaction at D



By using Pythagoras theorem;

$$D = \sqrt{(177.8)^2 + (49.02)^2}$$
$$\therefore \quad D = 184.4\text{N}$$
$$\theta = \tan^{-1}\frac{49.05}{177.8} = 15.4°$$

Reaction at B (on member BCD)



By using Pythagoras theorem;

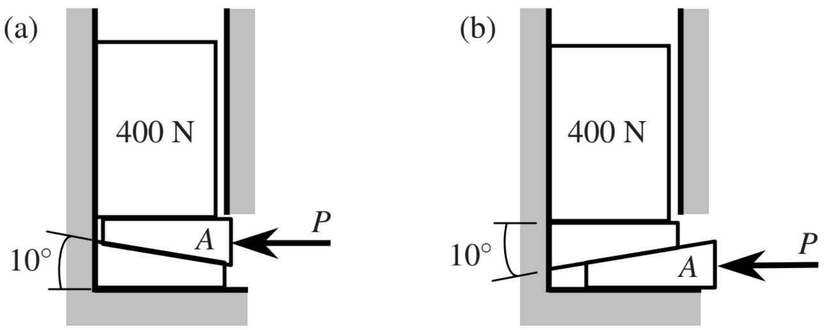
$$B = \sqrt{(196.2)^2 + (49.02)^2}$$
$$\therefore \quad B = 202.2\text{N}$$
$$\alpha = \tan^{-1}\frac{49.05}{196.2} = 14.04°$$

11.3 Wedges

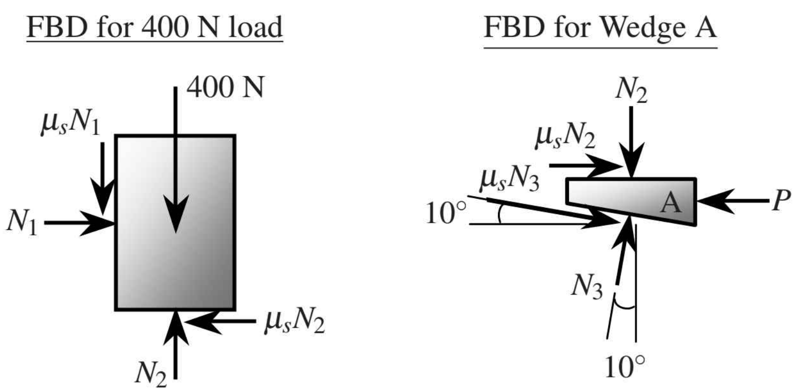
Wedges are simple machines utilised to raise/lower or maintain the position a heavy load. Load is raised or lowered by applying a force (usually much smaller than the load) at the wedge. With a suitable slant, friction between contacting surfaces will maintain the position of the wedge. Wedges are suitable where only small differences on the position of a heavy load are needed. Strategy for solutions to problems are similar to Category 4 friction problems.

Always make sure of the direction of friction force when drawing the free body diagram (FBD).

■ **Example 11.7** Determine the minimum force P so that the 400 N load is on the verge of moving up for both cases (a) and (b). The coefficient of static friction, $\mu_s = 0.25$.



Solution:
Case (a)



Observations:

- Draw the FBD of the wedge first to determine the correct direction of the friction force.
- For the same force, make sure that the direction is opposite on different surfaces of contact. For example, N_2 is (↑) for the 400 N load FBD and (↓) for the wedge

A FBD.

- Analysis is based on equilibrium of particles (without taking moments)

For 400 N load:

$$(+\rightarrow)\sum F_x = 0$$
$$N_1 - \mu_s N_2 = 0$$
$$N_1 - (0.25)N_2 = 0$$

$$(+\uparrow)\sum F_y = 0$$
$$N_2 - \mu_s N_1 - 400 = 0$$
$$N_2 - (0.25)N_1 - 400 = 0$$

solve simultaneously to obtain
$N_1 = 106.7$ N and $N_2 = 426.7$ N

For Wedge A:

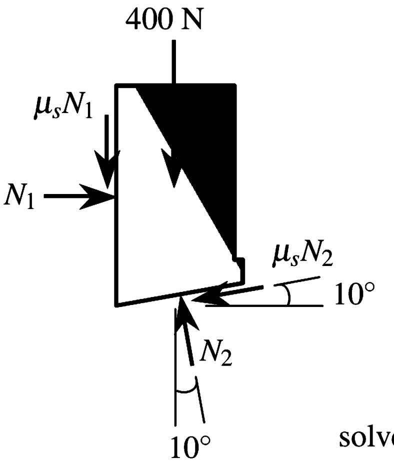
$$(+\rightarrow)\sum F_x = 0$$
$$\mu_s N_2 + \mu_s N_3 \cos 10^\circ + N_3 \sin 10^\circ - P = 0$$
$$(0.25)N_2 + (0.25)N_3 \cos 10^\circ + N_3 \sin 10^\circ - P = 0$$
$$(0.25)N_2 + (0.42)N_3 - P = 0$$
$$106.7 + (0.42)N_3 - P = 0 \cdots (1)$$

$$(+\uparrow)\sum F_y = 0$$
$$N_3 \cos 10^\circ - \mu_s N_3 \sin 10^\circ - N_2 = 0$$
$$(0.985)N_3 - (0.043)N_3 - 426.7 = 0$$
$$\therefore N_3 = 453 \text{ N}$$

by substituting N_3 into (1) to obtain $P = 297$ N

Case (b)

FBD for 400 N load



Taking the summation of forces along x and y axes yields;

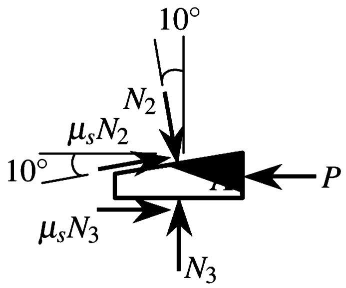
$$(+\rightarrow)\sum F_x = 0$$
$$N_1 - \mu_s N_2 \cos 10^\circ - N_2 \sin 10^\circ = 0$$
$$N_1 - (0.25) N_2 \cos 10^\circ - N_2 \sin 10^\circ = 0$$
$$N_1 - (0.42) N_2 = 0$$

$$(+\uparrow)\sum F_y = 0$$
$$N_2 \cos 10^\circ - \mu_s N_2 \sin 10^\circ - \mu_s N_1 - 400 = 0$$
$$(0.941) N_2 - (0.25) N_1 - 400 = 0$$

solve simultaneously to obtain
$N_1 = 201$ N and $N_2 = 478.5$ N

Taking the summation of forces along x and y axes yields;

FBD for Wedge A



$$(+\rightarrow)\sum F_x = 0$$
$$\mu_s N_3 + \mu_s N_2 \cos 10^\circ + N_2 \sin 10^\circ - P = 0$$
$$(0.25) N_3 + (0.25) N_2 \cos 10^\circ + N_2 \sin 10^\circ - P = 0$$
$$(0.25) N_3 + (0.42) N_2 - P = 0$$
$$(0.25) N_3 + 201 - P = 0 \cdots (1)$$

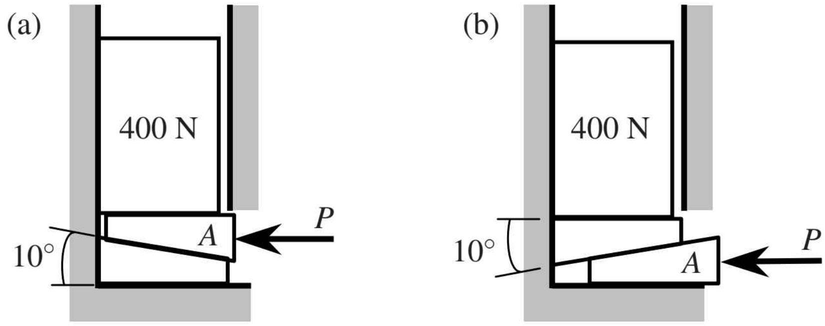
$$(+\uparrow)\sum F_y = 0$$
$$N_3 + \mu_s N_2 \sin 10^\circ - N_2 \cos 10^\circ = 0$$
$$N_3 + (0.25) N_2 \sin 10^\circ - N_2 \cos 10^\circ = 0$$
$$N_3 - (0.941) N_2 = N_3 - (450.3) = 0$$
$$N_3 = 450.3 \text{ N}$$

substituting N_3 into (1) to obtain $P = 313.6$ N

The answer reveals that the wedge arrangement in (a) is more suitable to maintain the position of the 400 N load because P is lower.

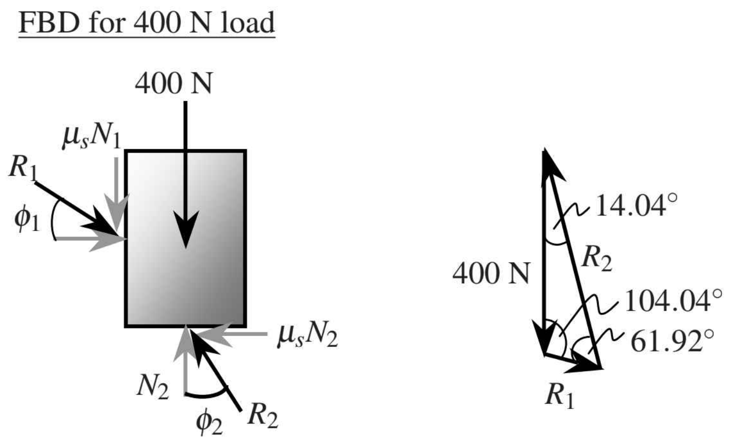
The normal solution showed above deals with many components of forces. As an alternative, the friction angle (ϕ) method can be used instead. This approach is shown by solving the same previous example.

■ **Example 11.8** Determine the minimum force P so that the 400 N load is on the verge of moving up for both cases (a) and (b). The coefficient of static friction, $\mu_s = 0.25$.



Solution:

Case (a)



from $\mu_s = \tan\phi$

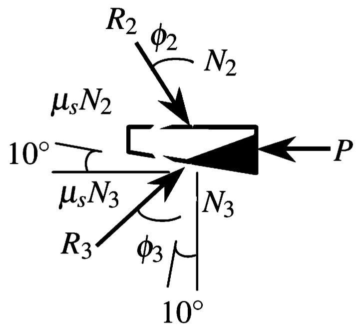
$$\therefore \phi = \phi_1 = \phi_2 = \tan^{-1}(0.25) = 14.04^\circ$$

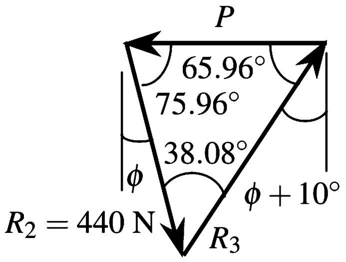
using the sine rule;

$$\frac{400}{\sin 61.92^\circ} = \frac{R_1}{\sin 14.04^\circ} = \frac{R_2}{\sin 104.04^\circ}$$

yields $R_1 = 110$ N and $R_2 = 440$ N

FBD for Wedge A





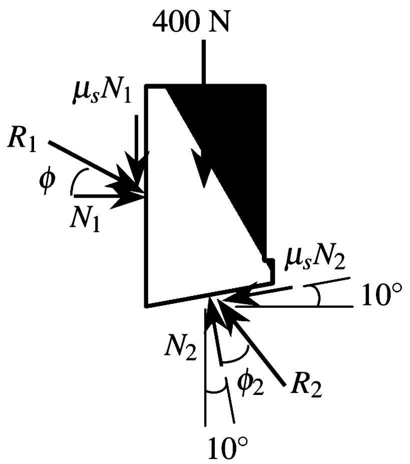
using the sine rule;

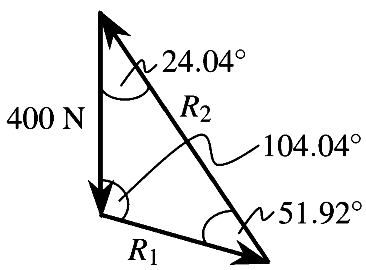
$$\frac{400}{\sin 65.96^\circ} = \frac{P}{\sin 38.08^\circ} = \frac{R_3}{\sin 75.96^\circ}$$

yields $P = 297$ N

Case (b)

FBD for 400 N load



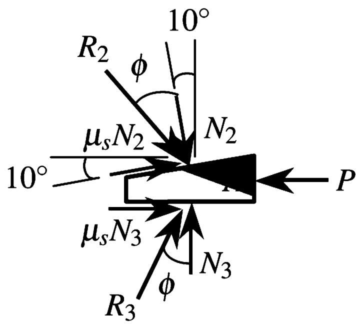


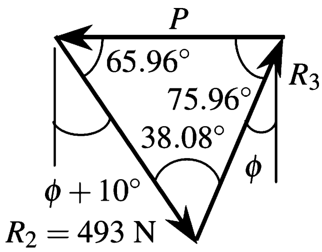
using the sine rule;

$$\frac{400}{\sin 51.92^\circ} = \frac{P}{\sin 24.04^\circ} = \frac{R_3}{\sin 104.04^\circ}$$

yields $R_1 = 207$ N and $R_2 = 493$ N

FBD for Wedge A





using the sine rule;

$$\frac{400}{\sin 75.96°} = \frac{P}{\sin 38.08°} = \frac{R_3}{\sin 65.96°}$$

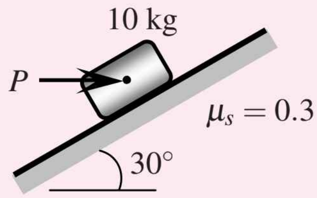
yields $P = 313.4$ N

11.4 Example questions

Exercise 11.1

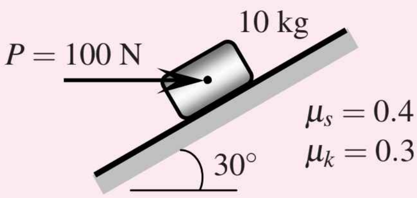
Determine the minimum force P required

a. to stop the block from sliding down the incline,
b. to start the block moving up the incline.



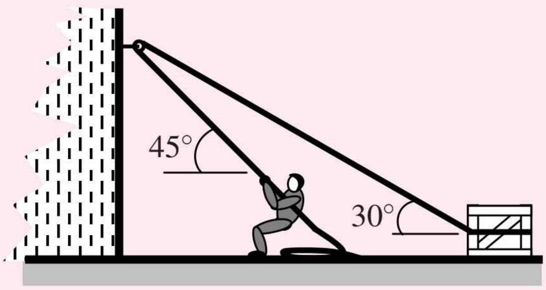
Exercise 11.2

Determine whether the block shown in the figure is in equilibrium, and find the value of the friction force.



Exercise 11.3

A 60 kg man tries to move the 100 kg mass using a cable and pulley system as shown. Determine whether he can move the block if the coefficients of static friction between the block and surface and between the ground and the man's shoes are 0.3 and 0.4 respectively.

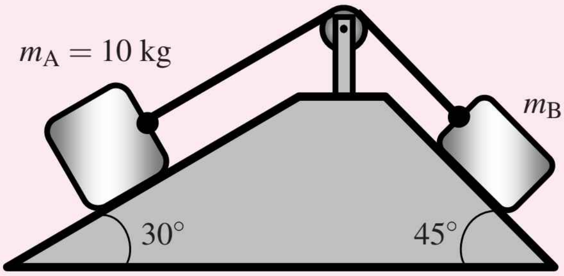


Exercise 11.4

Two blocks, a 10 kg mass, m_A and m_B are connected by cable on the incline planes as shown. If the coefficient of static friction between all contacting surfaces, $\mu_s = 0.3$, determine the mass m_B such that

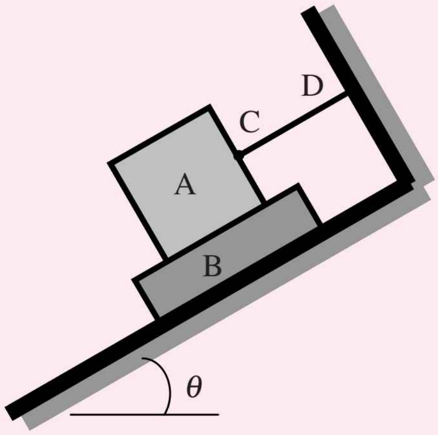
a. mass m_A is on the verge of moving up the 30° incline.
b. mass m_A is on the verge of moving down the 30° incline.

Answer:
(a) $m_B = 15.34$ kg, (b) $m_B = 2.61$ kg



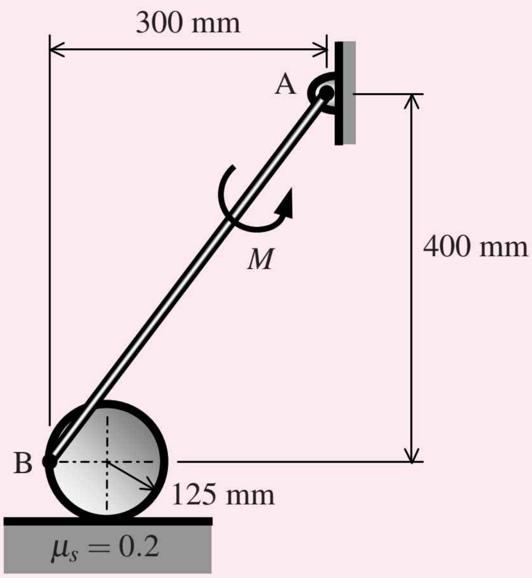
Exercise 11.5

The figure shows a 150 kg mass A is positioned on top of the 100 kg mass B. Determine the maximum angle θ and the tension in cable CD so that the system is still in equilibrium. The coefficient of static friction between all contacting surfaces, $\mu_s = 0.25$.



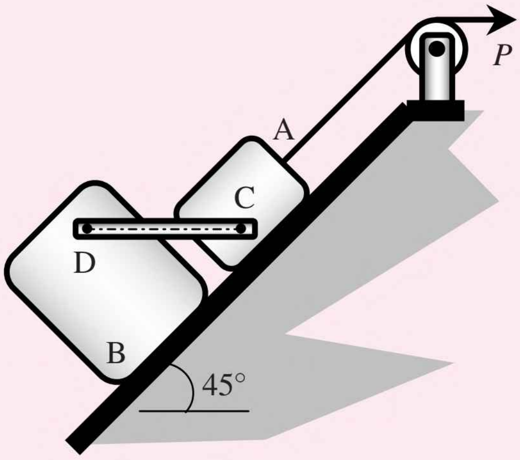
Exercise 11.6

Moment $M = 50$ Nm acting on member AB controls the movement of the 45 kg disc as shown. Determine the friction force between the surface and the disc.



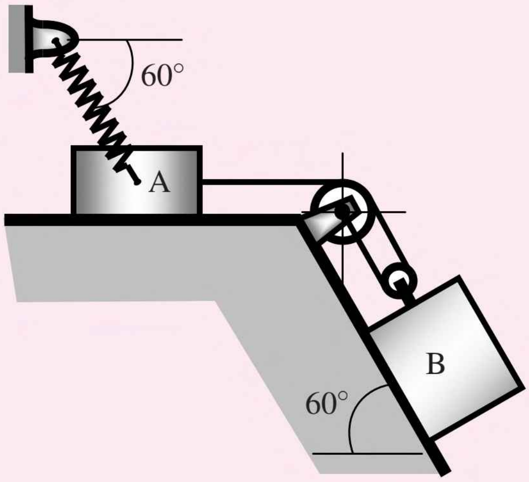
Exercise 11.7

The 180 kg mass A and 135 kg mass B are connected by the horizontal member CD. The coefficients of static friction (μ_s) between the surface and masses A and B are 0.3 and 0.5 respectively. Determine the smallest force P to stop the masses from sliding down.



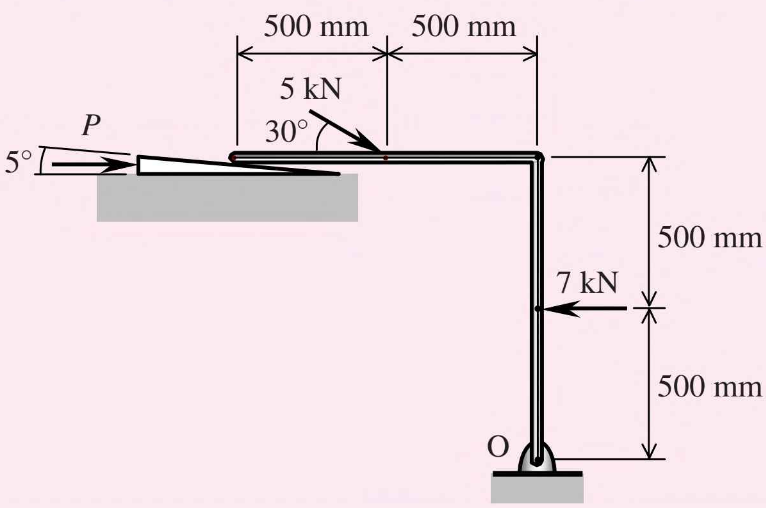
Exercise 11.8

The figure shows two masses A and B, 25 kg and 50 kg respectively. The coefficient of static friction between mass B and the surface is twice to that between mass A and the surface. Determine these values if the minimum tension in the spring to stop the masses from slipping is 120 N.



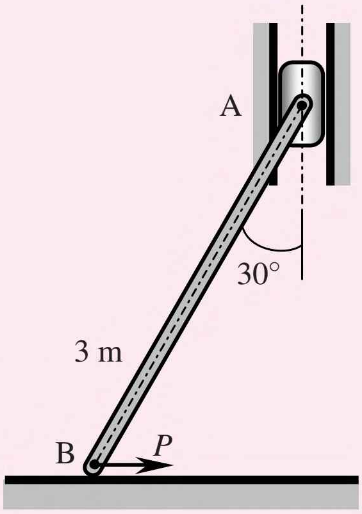
Exercise 11.9

A wedge is used to level member OAB as shown in the figure. Determine the force P that must be applied to start moving the wedge to the right. Given $\mu_s = 0.3$ for all contacting surfaces and neglect the size and weight of the wedge.



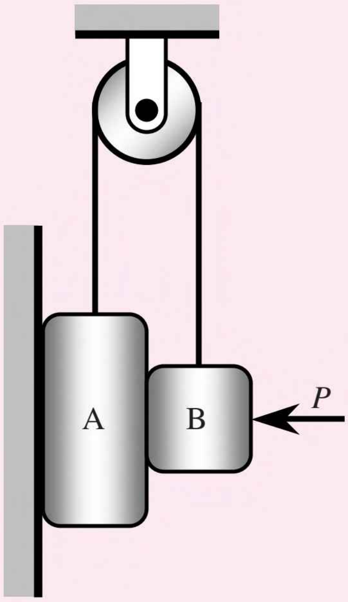
Exercise 11.10

The figure shows a 20 kg rod AB used to maintain the position of a 20 kg block A with a horizontal force P at end B. Knowing that the coefficient of static friction $\mu_s = 0.3$, determine the minimum force P required to stop block A from sliding down.



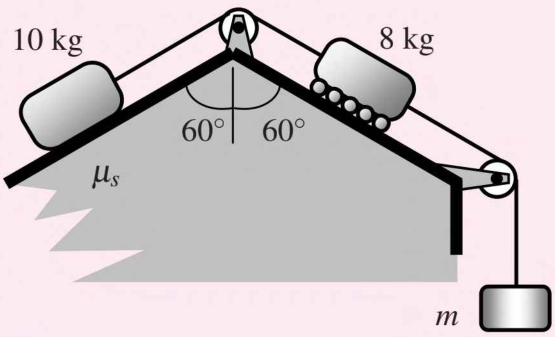
Exercise 11.11

A 14.4 kg mass block A and 7.2 kg mass block B are connected with a cable through a smooth pulley as shown. If the coefficient of static friction $\mu_s = 0.12$ for all contacting surfaces, determine the smallest force P to maintain equilibrium. Hence, determine the tension in the cable.



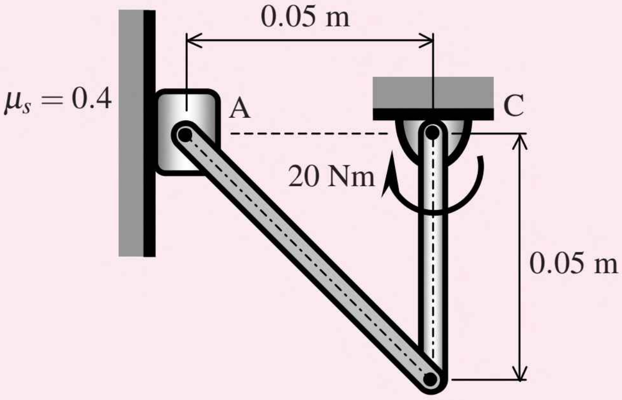
Exercise 11.12

a. Determine the mass m required to start the 10 kg block moving up the incline. The coefficient of static friction between the block and the surface is $\mu_s = 0.3$.
b. Determine the minimum value of μ_s to maintain equilibrium if $m = 0$.



Exercise 11.13

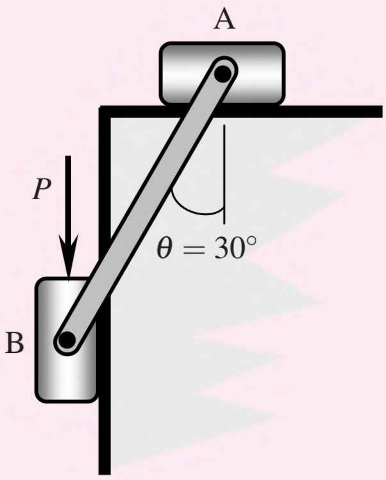
Determine the range of mass at A to maintain equilibrium.



Exercise 11.14

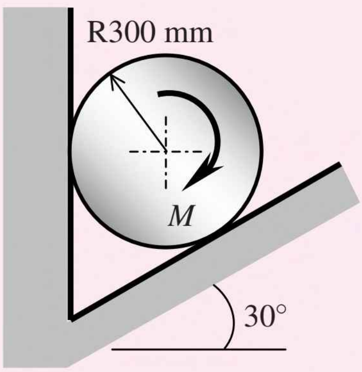
The two 10 kg blocks, A and B are connected by member AB as shown. The coefficient of static friction $\mu_s = 0.3$ for all contacting surfaces.

a. Show that the system is in equilibrium when $P = 0$ N,
b. Determine the maximum value of P to maintain equilibrium.



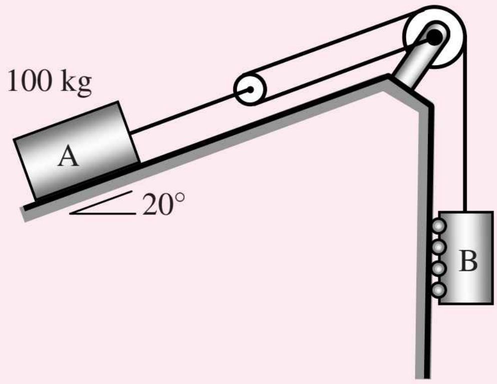
Exercise 11.15

A 100 kg cylinder is positioned against a vertical wall and a 30° slope as shown in the figure. If the coefficient of static friction for all contacting surfaces is $\mu_s = 0.5$, calculate the minimum couple M required to rotate the cylinder.



Exercise 11.16

Determine the range of values for m_B so that the 100 kg mass m_A is in equilibrium. The coefficient of static friction μ_s between block A and the inclination is 0.3. Answer: 3 kg $\leqslant m_\mathrm{B} \leqslant$ 31.2 kg



12 — Analysis of structures

12.1 Introduction

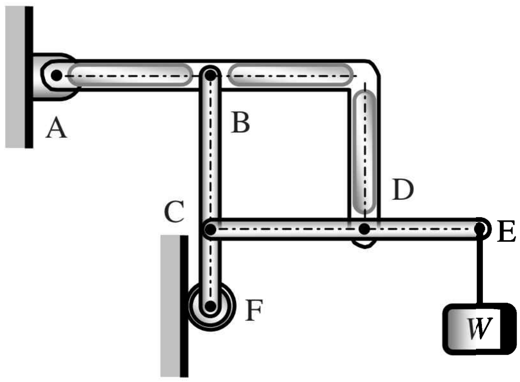
Prior to this topic, equilibrium of particles and rigid bodies have been extensively analysed, where attention is focused on external forces acting on the body as a whole. An engineering structure is a system consisting of inter-connected members, built to support load or transmit forces applied onto it, and must be designed to safely contain the applied forces.

Consequently, each and every member of the rigid body must also be capable to sustain part of the load acting on it. To ascertain this, members forming the rigid body have to be disconnected or separated so that an equilibrium analysis for each member can be carried out.

The forces that exist due to disconnection or separation of the members are called internal forces. The analysis conducted are based on Newton's third law which states that for every action there is an equal opposite reaction. The strategy for analysis is better illustrated by an example.

■ Example 12.1

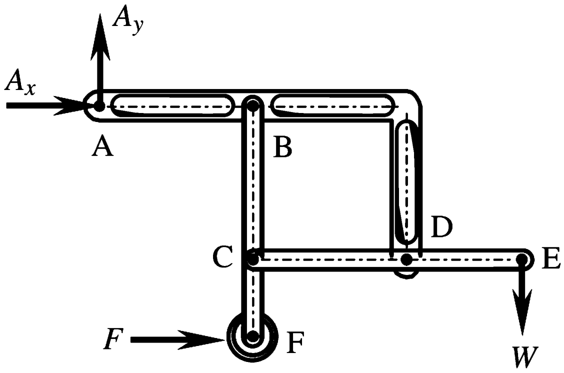
A three member mechanism pin jointed at A and roller at F is supporting load W at end E as shown. Determine all force components acting on every member.



Solution:
The first step is to draw the FBD of the whole rigid body.

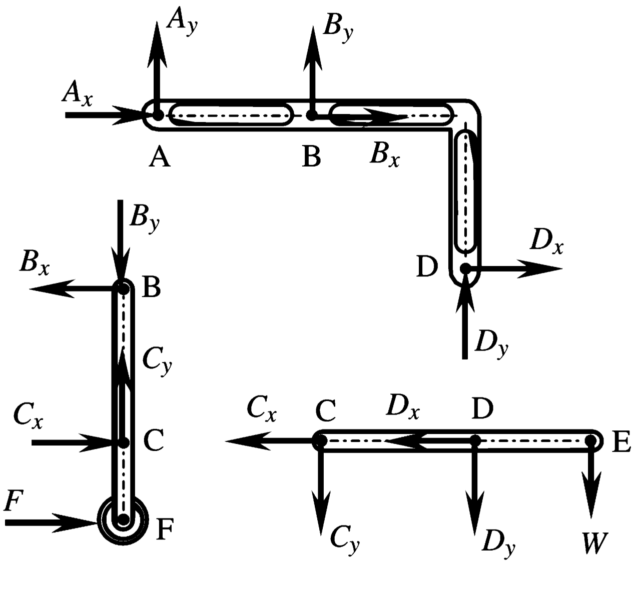
The values of A_x, A_y and F are determined by using the equilibrium equations;

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M_k = 0 \quad \text{(where k is any point)}$$



To determine the internal forces (which are reactions at the joints maintaining the shape of the structure) all members have to be disconnected or separated so that analysis on each member can be done.

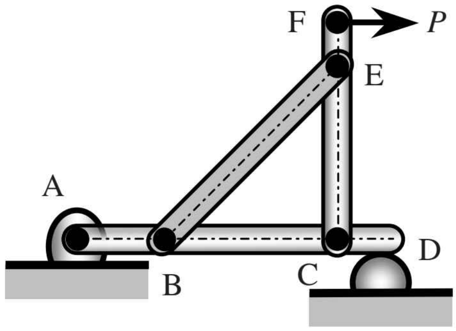
The next step is to draw the FBD of each member as shown below. It is very important to notice that when an assumption of the direction of a force (reaction) component at a point on a member, then the same force component acting on a different body must be opposite in direction. For example, B_x (the horizontal force component of point B) is assumed to the right on member ABD, then it must be assumed to the left on member BCF. If calculations were done on member ABD and giving a positive answer, then the assumed direction is correct for both members. If B_x is found to be negative, then the direction is to the left for member ABD and to the right for member BCF.



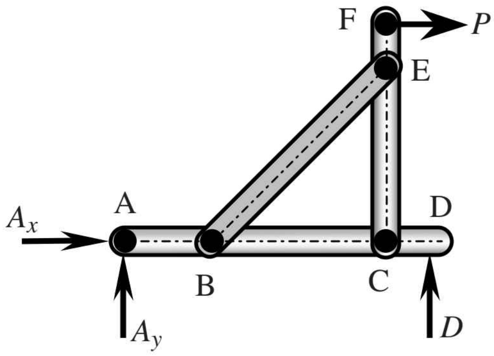
If a rigid body is in equilibrium, then each member forming the rigid body must also be in equilibrium. Now, similar to equilibrium of rigid bodies, calculations are done using the equilibrium equations ($\sum F_x = 0$, $\sum F_y = 0$ and $\sum M_k = 0$, where k is any point) for each member. If there is a two-force member among the disconnected members, remember that there is only one unknown (the magnitude) because the line of action of the reaction is known, as shown in the following example.

■ **Example 12.2**

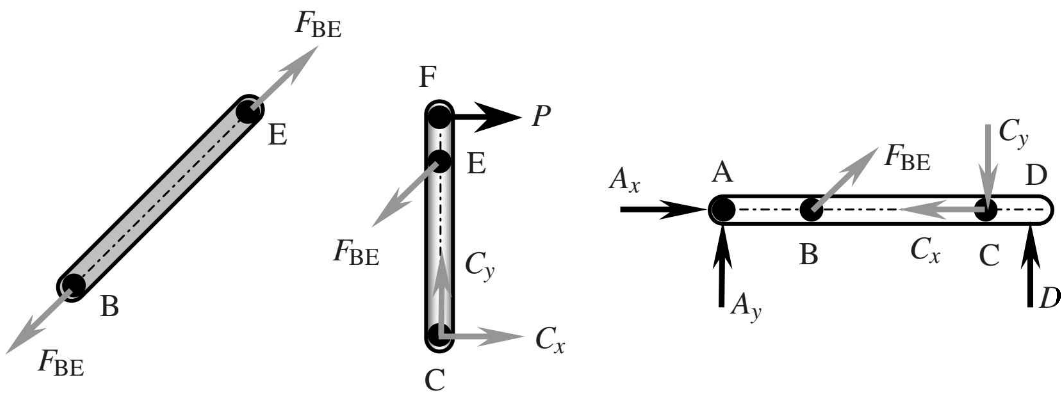
Force P is applied at point F on the frame as shown. Determine the components of forces acting on all members.



Solution:
Obtain the overall FBD;



Separate and draw the FBD of each member;



Notice that BE is a two-force member, which involves only one unknown, F_{BE}. Force F_{BE} is assumed to be in tension where the reaction is acting outwards of the point. A positive answer will mean that the assumed direction is correct while a negative answer will mean that the force is acting towards the point, which means that member BE is in compression. Sometimes, a correct assumption can be made. Observe that by taking moments about point C on FBD of member FEC, the force P will produce a clockwise moment. To maintain equilibrium, F_{BE} must produce a counter clockwise moment, which means that F_{BE} must act outwards from point E (the assumption shown in the FBD is correct). A correct assumption will remarkably simplify calculations as there is no requirement to change the direction of a particular force on other member.

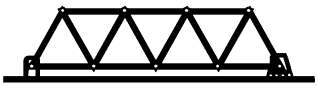
The following sections will cover the analysis of structures which include

- Analysis of trusses
- Analysis of frames and machines

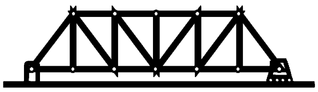
12.2 Analysis of trusses

The truss is one of the most popular type of engineering structures due to its efficiency in solving engineering problems practically and economically. Trusses are usually employed in designing bridges and buildings. Figure below shows some typical trusses.

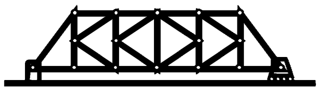
Typical bridge trusses



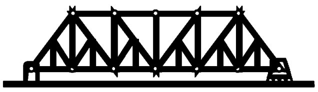
Warren



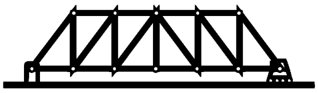
Pratt



K truss

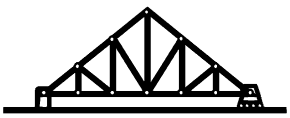


Baltimore

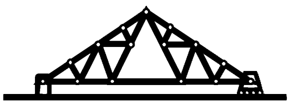


Howe

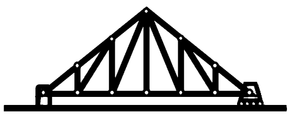
Typical roof trusses



Howe

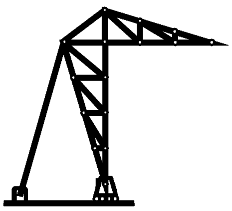


Fink

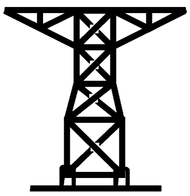


Pratt

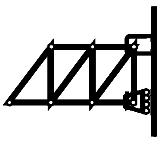
Stadium truss



'High voltage' tower truss



'Cantilever' truss



Trusses are made up of members with end connections only. Every member is designed to support load in the same plane, the reason why most analysis of trusses are done in two dimensions (one plane). All forces acting on a truss are considered to act at the connections (at the joints). Mass of each member is also considered to act at the joints. Even though in reality connection between members are either welded or bolted, the members are considered to be pin jointed. In short, all members used in the construction of a truss are two-force members, and a truss is considered as an integration of a group of joints with a collection of two-force members. Every member can be either in tension or compression. On the whole, solutions of problems involving trusses are based on the third Newton's law, as shown in the diagram below. Notice that in order to maintain equilibrium, unknown forces for each member is opposite in direction at matching joints. Also observe that all unknown forces are assumed to be in tension (directing outwards from the joints).

Diagram for problem

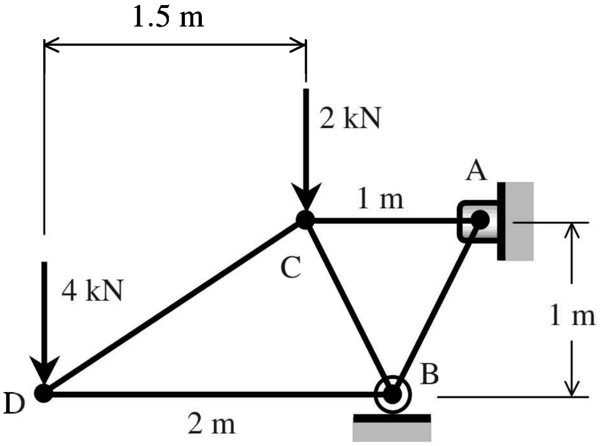
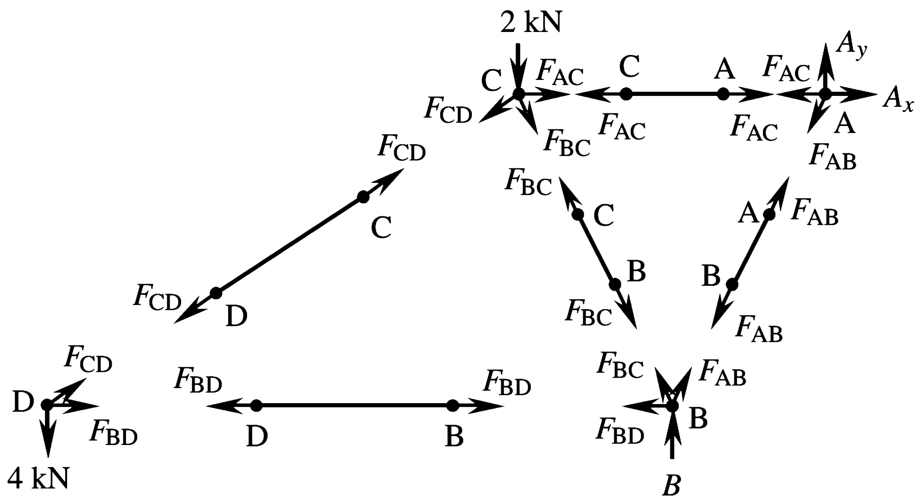


Diagram for analysis



There are two common methods of solving problems involving trusses, namely

- Method of joints
- Method of sections

12.2.1 Analysis of trusses by method of joints

Solution strategy:

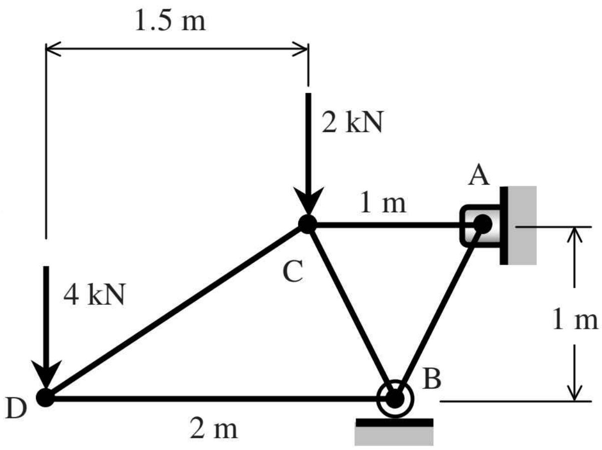
1. Draw the overall FBD and solve for reactions if applicable.
2. Choose a joint for analysis, where;
   a. there are a maximum of 2 unknown forces (magnitude),
   b. there is a minimum of one known force (magnitude).

   Assume all unknown forces to be in tension (directing outwards of the point). A positive answer means that the assumption is correct and the member is in tension. A negative answer means that the assumption is wrong and the member is actually in compression. Use the equilibrium of particles equations, $F_x = 0$ and $F_y = 0$ to solve. The trigonometry method can also be used.
3. Repeat step 2 for other joints until all required forces are obtained. If a positive answer is achieved for a force then the force must also be in tension (directing outwards) from the next connecting joint. If the answer is negative, then there are two possible alternative solutions;
   a. if referring to the original joint, the force direction is maintained while the magnitude is negative in sign. OR
   b. draw a new FBD putting the force directing to the joint while maintaining a positive magnitude.

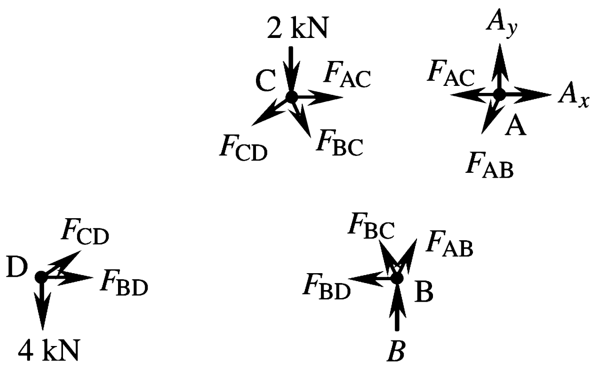
The solution strategy is illustrated by solving the following example.

■ Example 12.3

Two forces are applied to the truss shown determine the forces in each member and state whether the member is in tension or compression.



Solution:
The FBD for each joint is given as;



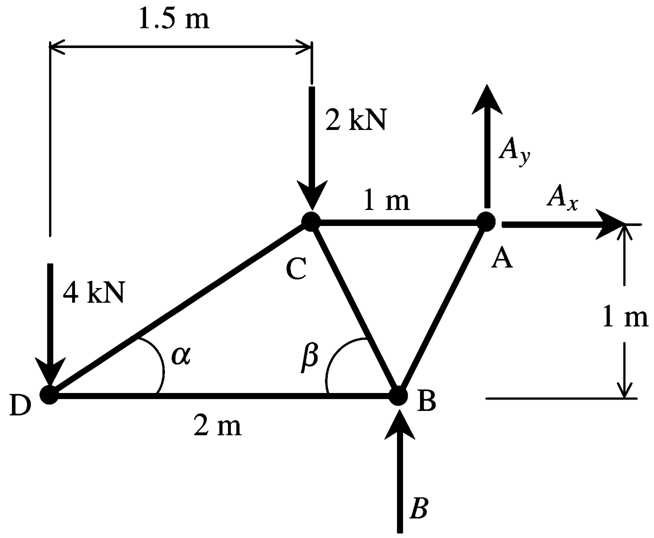
Observations:

- at joint A → 4 unknown forces.
- at joint B → 4 unknown forces.
- at joint C → 3 unknown and 1 known forces.
- at joint D → 2 unknown and 1 known forces.

It becomes clear that analysis can only begin at joint D followed by C, then either A or B. However, a more common variation is to first determine external reactions by solving the overall FBD. Observations will then be;

- at joint A → 2 unknown and 2 known forces.
- at joint B → 3 unknown and 1 known forces.
- at joint C → 3 unknown and 1 known forces.
- at joint D → 2 unknown and 1 known forces.

Analysis can begin at joint A or D. Overall FBD is given as;



$$(+\circlearrowleft)\sum M_A = 0$$

$$4 \text{ kN}(2.5 \text{ m}) + 2 \text{ kN}(1 \text{ m}) - B(0.5 \text{ m}) = 0$$

$$\therefore B = 24 \text{ kN}$$

$$(+\rightarrow)\sum F_x = 0$$

$$\therefore A_x = 0$$

$$(+\uparrow)\sum F_y = 0$$

$$A_y + B - 4 \text{ kN} - 2 \text{ kN} = 0$$

$$A_y + 24 \text{ kN} - 4 \text{ kN} - 2 \text{ kN} = 0$$

$$A_y = -18 \text{ kN}$$

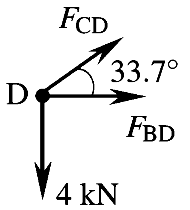
$$\therefore A_y = 18 \text{ kN}(\downarrow)$$

determine angles α and β.

$$\alpha = \tan^{-1}\left(\frac{1}{1.5}\right) = 33.7^\circ \quad \text{and} \quad \beta = \tan^{-1}\left(\frac{1}{0.5}\right) = 63.4^\circ$$

Analysis of trusses by method of joints;

<u>Joint D</u>



$$(+\uparrow)\sum F_y = 0$$

$$F_{CD}\sin 33.7^\circ - 4 \text{ kN} = 0$$

$$\therefore F_{CD} = 7.21 \text{ kN} = 0(\text{tension})$$

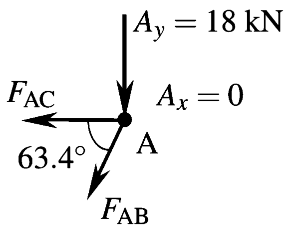
$$(+\rightarrow)\sum F_x = 0$$

$$F_{CD}\cos 33.7^\circ + F_{BD} = 0$$

$$F_{BD} = -6 \text{ kN}$$

$$\therefore F_{BD} = 6 \text{ kN}(\text{compression})$$

<u>Joint A</u>



$$(+\uparrow)\sum F_y = 0$$

$$-18 \text{ kN} - F_{AB}\sin 63.4^\circ = 0$$

$$F_{AB} = -20.1 \text{ kN}$$

$$\therefore F_{AB} = -20.1 \text{ kN}(\text{compression})$$

$$(+\rightarrow)\sum F_x = 0$$

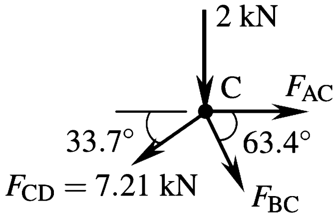
$$-F_{AB}\cos 63.4^\circ + F_{AC} = 0$$

$$-(-20.1 \text{ kN})\cos 63.4^\circ - F_{AC} = 0$$

$$\therefore F_{AC} = 9 \text{ kN}(\text{tension})$$

The next analysis conducted can be on either joint B or C.

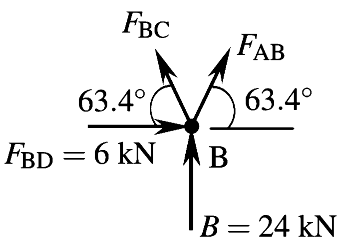
Joint C



$$(+\uparrow)\sum F_y = 0$$
$$-2\text{ kN} - 7.21\sin 33.7° - F_{BC}\sin 63.4° = 0$$
$$F_{BC} = -6.71\text{ kN}$$
$$\therefore F_{BC} = 6.71\text{ kN(compression)}$$
$$(+\rightarrow)\sum F_x = 0$$
$$F_{AC} - 7.21\cos 33.7° + F_{BC}\cos 63.4° = 0$$
$$F_{AC} - 7.21\cos 33.7° + (-6.7)\cos 63.4° = 0$$
$$\therefore F_{AC} = 9\text{ kN(tension)}$$

Observe that F_{AC} is the same as previously, when analyse at joint A.
If pin B is chosen;

Joint B



$$(+\rightarrow)\sum F_x = 0$$
$$6 + F_{AB}\cos 63.4° - F_{BC}\cos 63.4° = 0$$
$$6 + 0.448F_{AB} - 0.448F_{BC} = 0$$
$$13.4 + F_{AB} - F_{BC} = 0\ldots(1)$$
$$(+\uparrow)\sum F_y = 0$$
$$24 + F_{AB}\sin 63.4° + F_{BC}\sin 63.4° = 0$$
$$24 + 0.894F_{AB} + 0.894F_{BC} = 0$$
$$26.8 + F_{AB} + F_{BC} = 0\ldots(2)$$

solving both equations (1) and (2) yield;

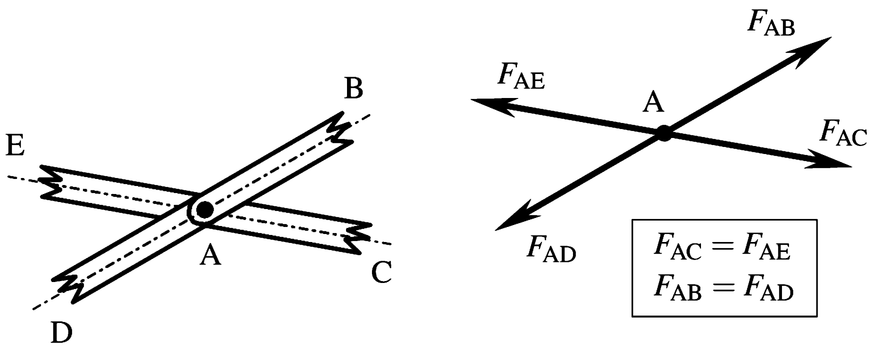
$$F_{AB} = -20.1\text{ kN} = 20.1\text{ kN(compression)}$$
$$F_{BC} = -6.7\text{ kN} = 6.7\text{ kN(compression)}$$

The answers are the same with previous calculations. This extra joint is usually used to verify earlier calculations.

12.2.2 Special loading conditions

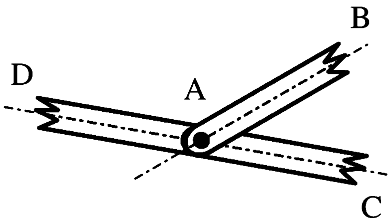
The special loading conditions are as follows:

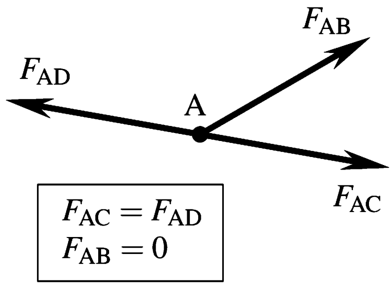
1. **4 forces/members with two line of actions.**



If member AE is found to be in tension (the force acting outwards from A), then member AC is also in tension. The same applies to members AB and AD.

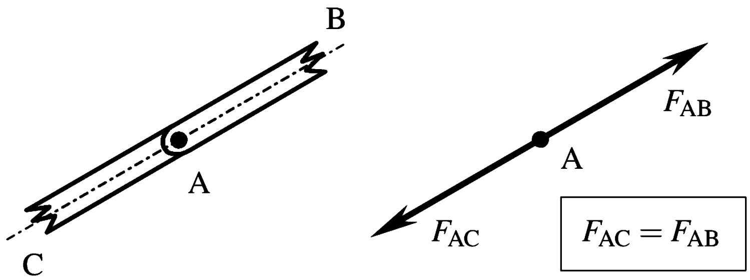
2. **3 forces/members with two of them on the same line of action.**





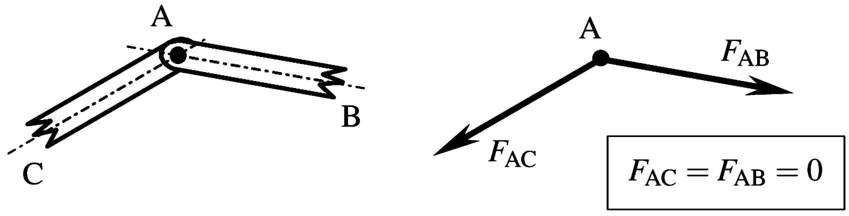
AB is known as zero force member.

3. **2 forces/members and both of them on the same line of action.**



If member AB is found to be in tension (the force acting outwards from A), then member AC is also in tension.

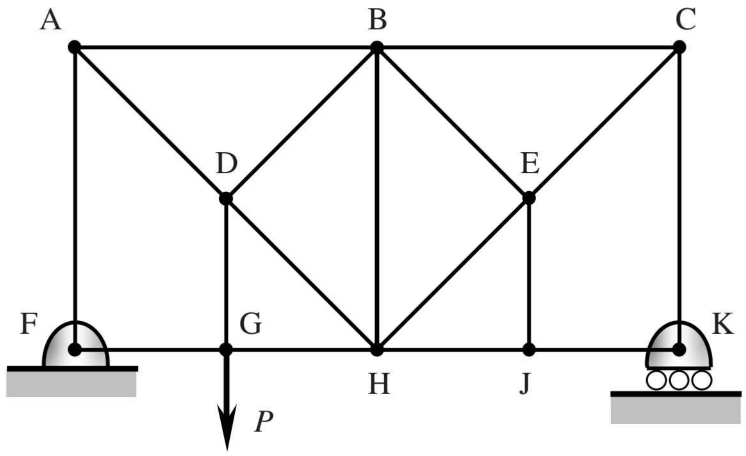
4. **2 forces/members and both of them not on the same line of action.**

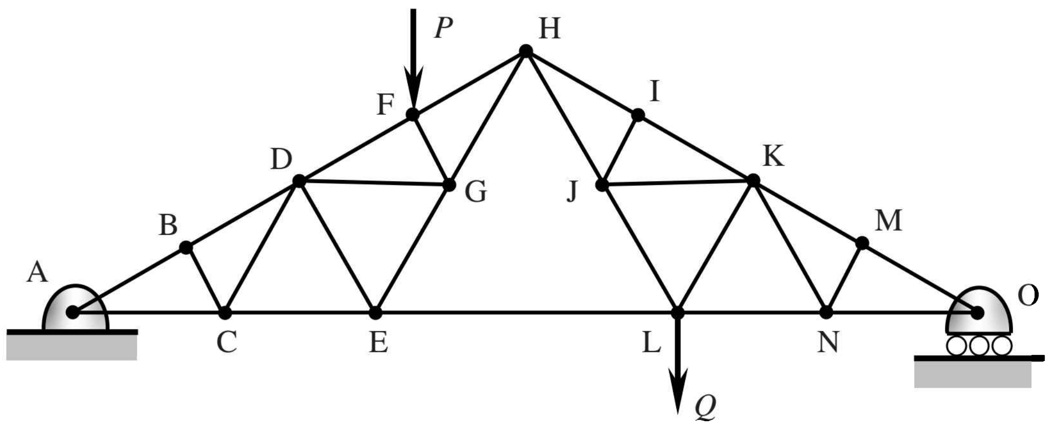


Both members, AB and AC are zero-force members.

Zero-force members do not actually mean that the member is not required. The member exist to compensate for different loading conditions. The member is also needed to preserve the shape of the truss.

■ **Example 12.4** Identify the zero-force members in the truss shown.





12.2.3 Analysis of trusses by method of sections

Usually, in problems involving trusses, the aim is to find only a few forces relating to some members of the truss. Even though the joint method is available, using this approach would involves long and tedious calculations. In this type of problems, it is more convenient to use the method of sections. The method of sections is based on the fact that if a body is in equilibrium, then any part of the body must also be in equilibrium.

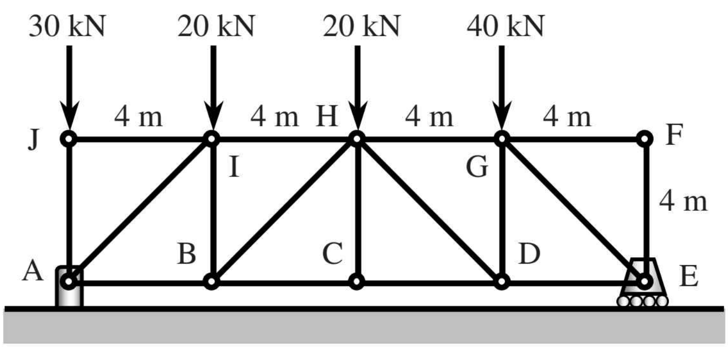
Strategy of analysis:

1. Draw the FBD and solve for reactions, if applicable.
2. Make a decision on sectioning the truss:
   - 'cutting' through required members (based on the problem).
   - since there are only three available equations, $\sum F_x = 0$, $\sum F_y = 0$ and $\sum M_A = 0$ (where A is any point), make sure that the 'cut' involves only three members.
3. When the truss is divided into two sections, choose the simpler one to solve. 'simpler' means:
   - the section with less forces.
   - where most of the forces involved are horizontal and/or vertical.
4. Assume all unknown forces to be in tension (directing outwards of the point). A positive answer means that the assumption is correct and the member is in tension. A negative answer means that the assumption is wrong and the member is actually in compression.
5. Solve using the three equilibrium equations, $\sum F_x = 0$, $\sum F_y = 0$ and $\sum M_A = 0$ (where A is any point). To reduce the risks of making mistakes, it is advisable to find one equation that will directly solve one unknown.

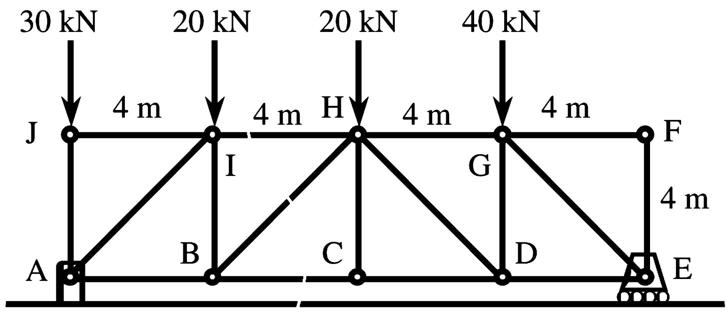
The solution strategy is better illustrated by following example.

■ Example 12.5

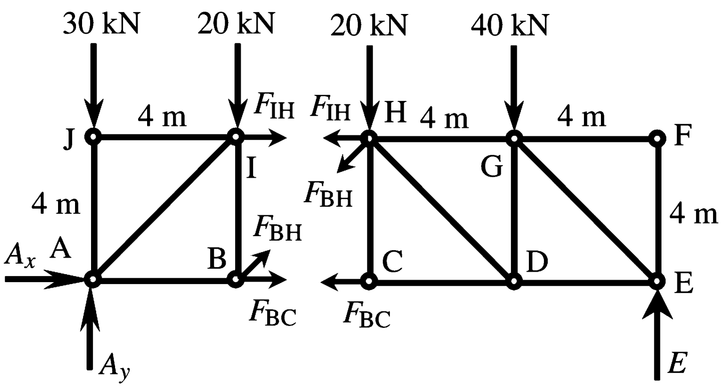
Determine the forces in members HI, HB and BC, and determine whether the members are in tension or compression.



Solution:
Draw a line crossing the members required in the question.

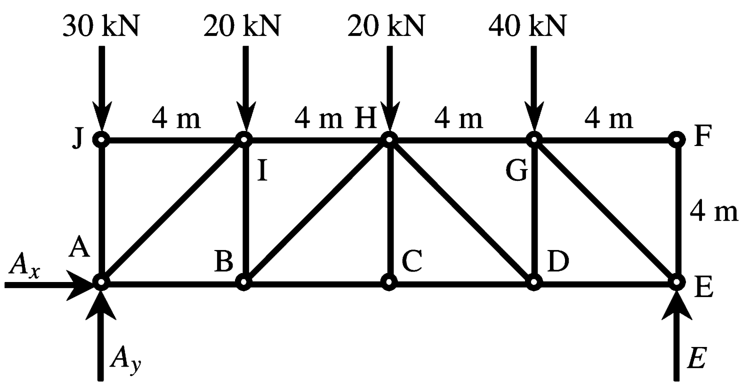


After 'cutting' step, the truss is divided into two sections.



In order to solve the left side, the reaction at A must be known. In order to solve the right side, the reaction at E must be known. Therefore, it is practical to solve the overall FBD first.

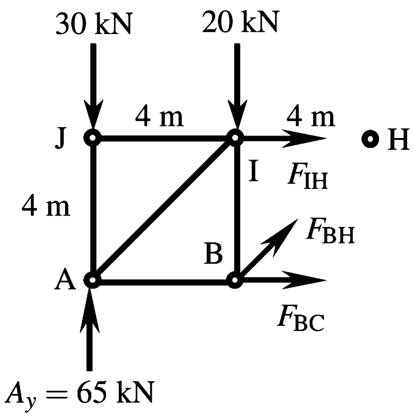
<u>Overall FBD</u>



$$(+\circlearrowright)\sum M_A = 0$$
$$20\text{ kN}(4\text{ m}) + 20\text{ kN}(8\text{ m}) + 40\text{ kN}(12\text{ m}) - E(16\text{ m}) = 0$$
$$\therefore E = 45\text{ kN}(\uparrow)$$
$$(+\rightarrow)\sum F_x = 0$$
$$\therefore A_x = 0$$
$$(+\uparrow)\sum F_y = 0$$
$$A_y - 30\text{ kN} - 20\text{ kN} - 20\text{ kN} - 40\text{ kN} + 45\text{ kN} = 0$$
$$\therefore A_y = 65\text{ kN}(\uparrow)$$

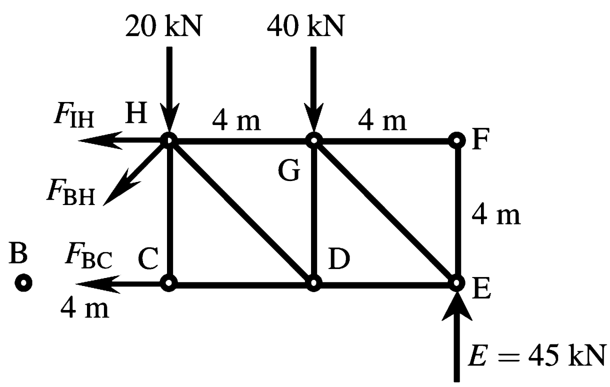
The next step is to choose the section to solve. Solutions for both sections are shown for this question. Notice that all unknown forces are assumed to be in tension (acting outwards of the points). To reduce the risks of making mistakes, it is advisable to find one equation that will directly solve one unknown.

Left hand side



$$(+\circlearrowright)\sum M_B = 0$$
$$F_{IH}(4\text{ m}) - 30\text{ kN}(4\text{ m}) + 65\text{ kN}(4\text{ m}) = 0$$
$$F_{IH} = -35\text{ kN}$$
$$\therefore F_{IH} = 35\text{ kN(compression)}$$
$$(+\rightarrow)\sum F_y = 0$$
$$F_{BH}\sin 45^\circ + 65\text{ kN} - 30\text{ kN} - 20\text{ kN} = 0$$
$$F_{BH} = -21.2\text{ kN}$$
$$\therefore F_{BH} = 21.2\text{ kN(compression)}$$
$$(+\circlearrowright)\sum M_H = 0$$
$$-F_{BC}(4\text{ m}) - 30\text{ kN}(8\text{ m}) - 20\text{ kN}(4\text{ m}) + 65\text{ kN}(8\text{ m}) = 0$$
$$F_{BC} = 50\text{ kN(tension)}$$
$$(+\rightarrow)\sum F_x = 0$$
$$F_{IH} + F_{BC} + F_{BH}\cos 45^\circ = 0$$
$$-35\text{ kN} + 50\text{ kN} + F_{BH}\cos 45^\circ = 0$$
$$F_{BH} = -21.2\text{ kN}$$
$$\therefore F_{BH} = 21.2\text{ kN(compression)}$$

Right hand side



$(+\circlearrowright)\sum M_H = 0$

$40\text{ kN}(4\text{ m}) - 45\text{ kN}(8\text{ m}) + F_{BC}(4\text{ m}) = 0$

$F_{BC} = 50\text{ kN(tension)}$

$(+\rightarrow)\sum F_y = 0$

$-F_{BH}\sin 45^\circ + 45\text{ kN} - 60\text{ kN} - 20\text{ kN} = 0$

$F_{BH} = -21.2\text{ kN}$

$\therefore F_{BH} = 21.2\text{ kN(compression)}$

$(+\circlearrowright)\sum M_B = 0$

$-F_{IH}(4\text{ m}) - 45\text{ kN}(12\text{ m}) + 20\text{ kN}(4\text{ m}) + 40\text{ kN}(8\text{ m}) = 0$

$F_{IH} = -35\text{ kN}$

$\therefore F_{IH} = 35\text{ kN(compression)}$

$(+\rightarrow)\sum F_x = 0$

$-F_{IH} - F_{BC} - F_{BH}\cos 45^\circ = 0$

$-(-35\text{ kN}) - 50\text{ kN} - F_{BH}\cos 45^\circ = 0$

$F_{BH} = -21.2\text{ kN}$

$\therefore F_{BH} = 21.2\text{ kN(compression)}$

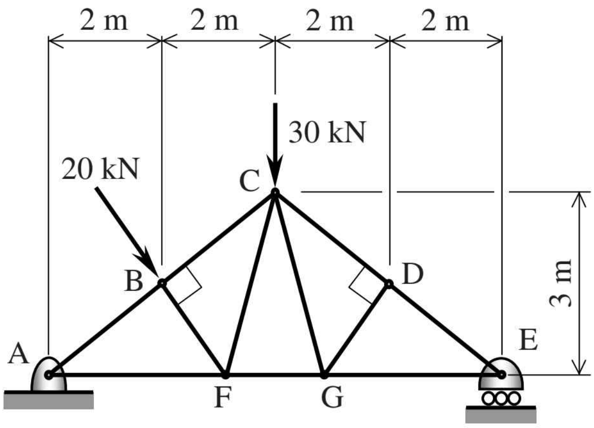
■ Example 12.6

State (if any) the zero–force members for the truss shown. Determine the force in members;

(a) CD and FG by using the methods of joints.

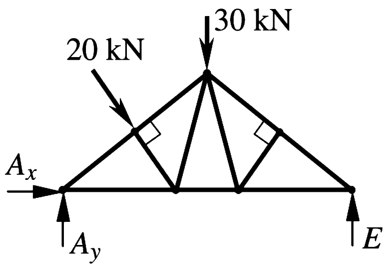
(b) CB, CF and FG by using the methods of sections.

State whether the members are in tension (T) or compression (C).



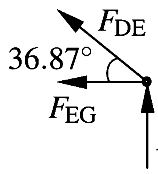
Solution:

The zero force members are DG and CG.
From overall FBD;



$$(+\circlearrowleft)M_A = 0$$
$$20(2.5) + 30(4) - E(8) = 0$$
$$E = 21.25 \text{ kN}(\uparrow)$$
$$(+\uparrow)F_y = 0$$
$$A_y + E - 20\cos 36.87^\circ - 30 = 0$$
$$A_y = 24.75 \text{ kN}(\uparrow)$$
$$(+\rightarrow)F_x = 0$$
$$A_x + 20\sin 36.87^\circ = 0$$
$$A_x = -12 \text{ kN}(\text{negative value}, \leftarrow)$$

(a) Using method of joint, start from joint E;



$$(+\uparrow)F_y = 0$$
$$21.25 + F_{DE}\sin 36.87^\circ = 0$$
$$F_{DE} = -35.42 \text{ kN}(\text{negative value, C})$$
$$(+\rightarrow)F_x = 0$$
$$-F_{EG} - F_{DE}\cos 36.87^\circ = 0$$
$$F_{EG} = -(-35.42)\cos 36.87^\circ$$
$$F_{EG} = 28.3 \text{ kN}(\text{positive value, T})$$

From joint D;

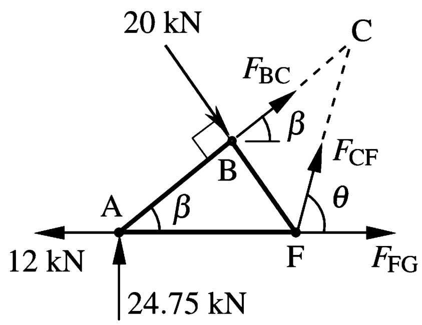
$$F_{CD} = F_{DE} = 35.42 \text{ kN}(C)$$

From joint G;

$$F_{FG} = F_{EG} = 28.3 \text{ kN}(T)$$

(b) Using method of section;

$$\beta = \tan^{-1}\left(\frac{1.5}{2}\right) = 36.87^\circ$$
$$\theta = 180^\circ - 2(53.13^\circ) = 73.74^\circ$$



$$(+\circlearrowleft)M_C = 0$$
$$24.75(4) + 12(3) - 20(2.5) - F_{FG}(3) = 0$$
$$F_{FG} = 28.3 \text{ kN}(\text{positive value, T})$$
$$(+\circlearrowleft)M_F = 0$$
$$24.75(3.125) + F_{BC}(1.875) = 0$$
$$F_{BC} = 41.25 \text{ kN}(\text{negative value, C})$$
$$(+\circlearrowleft)M_A = 0$$
$$20(2.5) - F_{CF}\sin 73.74^\circ(3.125) = 0$$
$$F_{CF} = 16.67 \text{ kN}(\text{positive value, T})$$

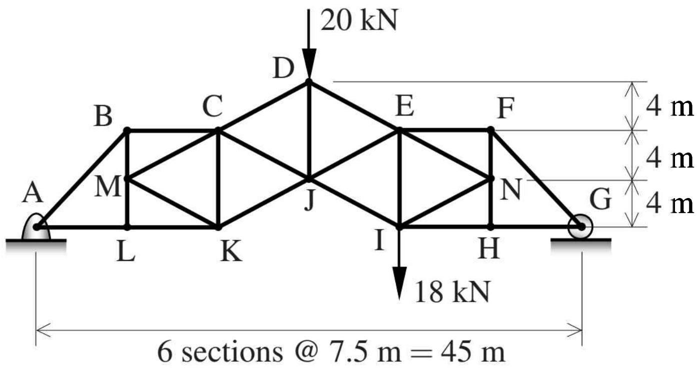
■ **Example 12.7**

Determine the force in members;

(a) AB and AL by using the methods of joints

(b) CD, CJ and KJ by using the methods of sections.

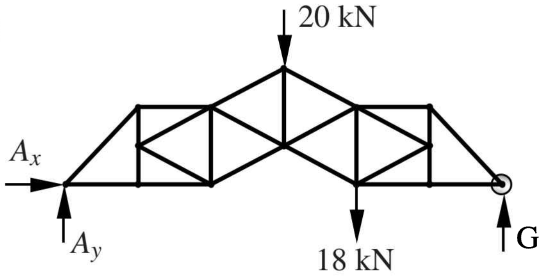
State whether the members are in tension (T) or compression (C).



Solution:

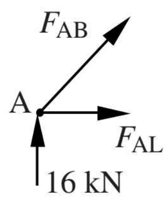
From overall FBD;

$$(+\circlearrowright) M_A = 0$$
$$20(22.5) + 18(30) - G(45) = 0$$
$$G = 22 \text{ kN(positive value, } \uparrow)$$
$$(+\uparrow) F_y = 0$$
$$A_y - 20 - 18 + 22 = 0$$
$$A_y = 16 \text{ kN(positive value, } \uparrow)$$
$$(+\rightarrow) F_x = 0$$
$$A_x = 0 \text{ kN}$$

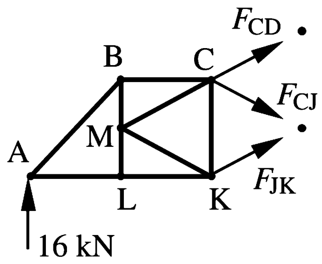


(a) At joint A;

$$(+\uparrow) F_y = 0$$
$$16 + F_{AB} \sin 46.85° = 0$$
$$F_{AB} = -21.93 \text{ kN(negative value, C)}$$
$$(+\rightarrow) F_x = 0$$
$$F_{AL} + F_{AB} \cos 46.85° = 0$$
$$F_{AL} = 15 \text{ kN(positive value, T)}$$



(b) Taking left hand side section;



$(+\circlearrowright) M_C = 0$

$16(15) - F_{JK}\cos 28.07°(8) = 0$

$F_{JK} = 34$ kN(positive value, T)

$(+\circlearrowright) M_J = 0$

$16(22.5) + F_{CD}\cos 28.07°(4) + F_{CD}\sin 28.07°(7.5) = 0$

$F_{CD} = -51$ kN(negative value, C)

$(+\rightarrow) F_x = 0$

$F_{CJ}\cos 28.07° + 34\cos 28.07° - 51\cos 28.07° = 0$

$F_{CJ} = 17$ kN(positive value, T)

$(+\uparrow) F_y = 0$

$16 - F_{CJ}\sin 28.07° + 34\sin 28.07° - 51\sin 28.07° = 0$

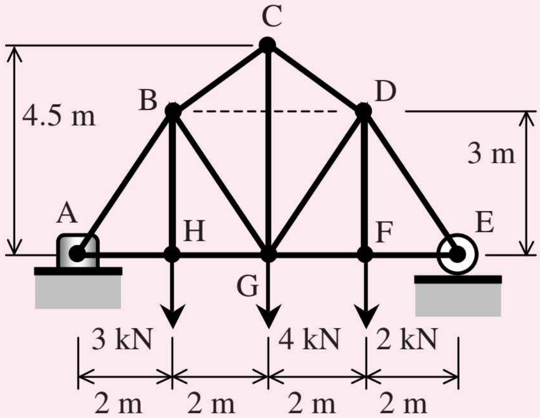
$F_{CJ} = 17$ kN(positive value, T)...checked!

12.3 Example questions (trusses)

Exercise 12.1

Determine the forces in following truss members and state whether they are in tension or compression;

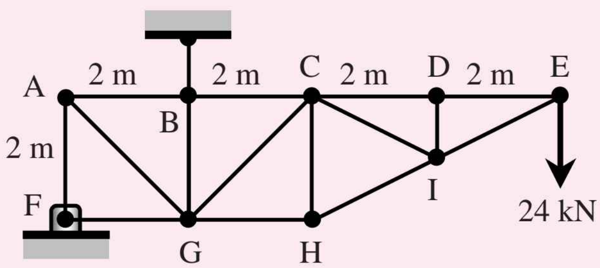
a. Truss members BC, BG and HG.
b. Truss members CD, DG and DF.



Answer: (a) $F_{BC} = 3.6$ kN (C), $F_{BG} = 0.5$ kN (C) and $F_{HG} = 3.17$ kN (T)

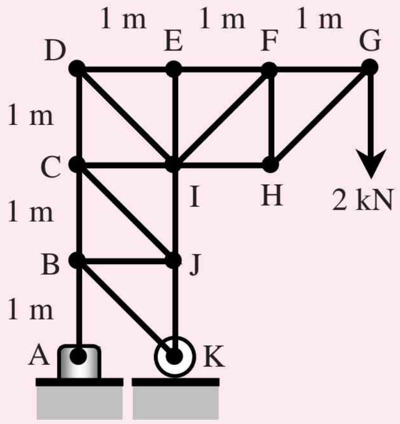
Exercise 12.2

Determine the force in members AG and CI of the truss shown. State whether the members are in tension or compression.



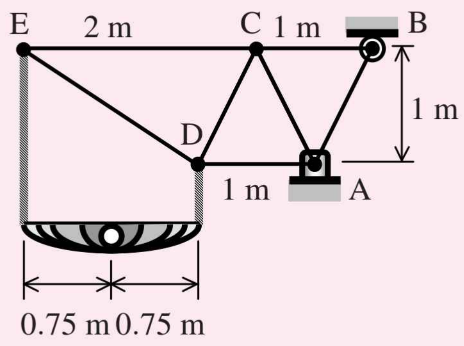
Exercise 12.3

Determine the force in members EF, IF and IH of the truss shown using
a. method of joints,
b. method of sections.
State whether the members are in tension or compression.



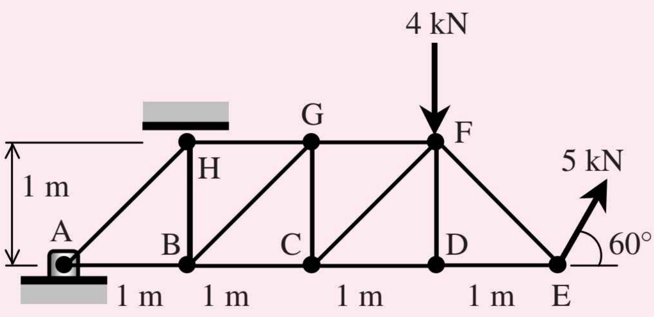
Exercise 12.4

A 120 kg mass is supported by cables at D and E. Determine the force in members CD and AD of the truss shown using
a. method of joints,
b. method of sections.
State whether the members are in tension or compression.



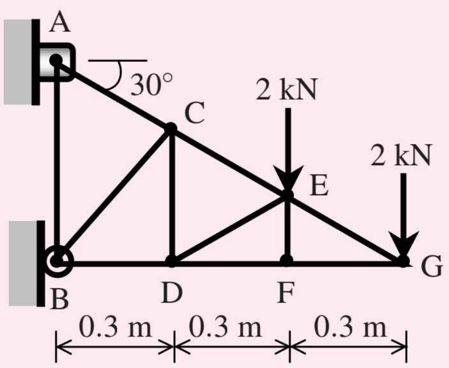
Exercise 12.5

Determine the force in members FG, CF and CD of the truss shown using
a. method of joints,
b. method of sections.
State whether the members are in tension or compression.



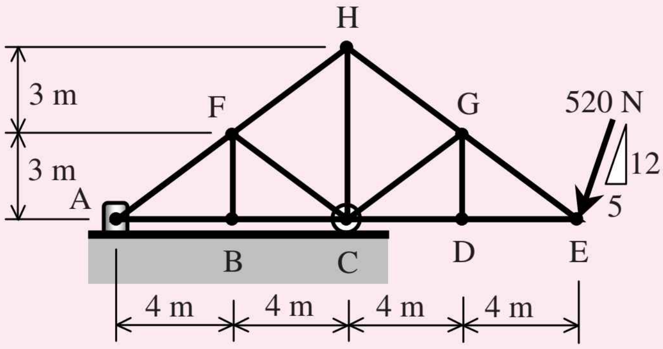
Exercise 12.6

Determine the force in members CE, DE and DF of the truss shown using
a. method of joints,
b. method of sections.
State whether the members are in tension or compression.



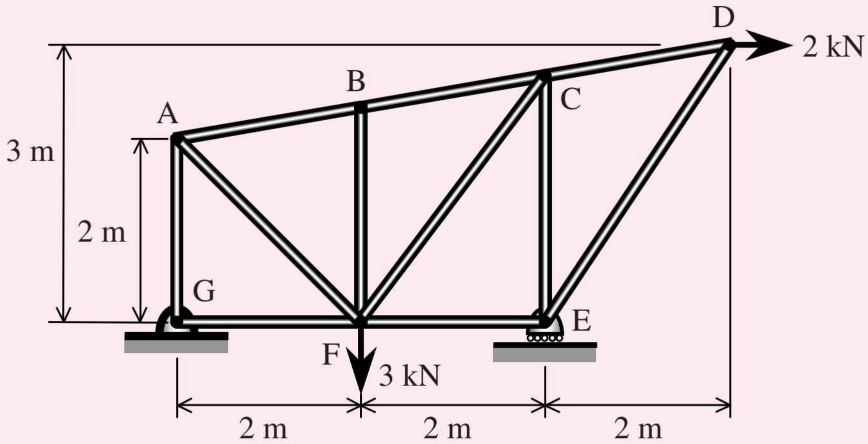
Exercise 12.7

Determine the force in members BF, CF and FH of the truss shown using
a. method of joints,
b. method of sections.
State whether the members are in tension or compression.



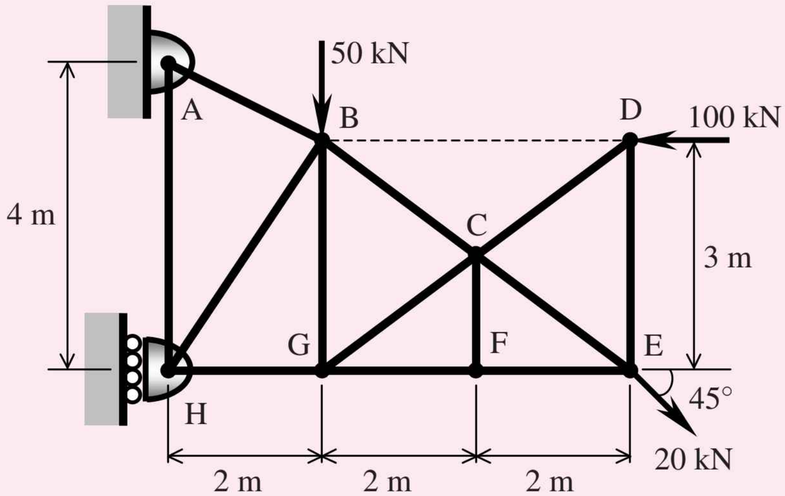
Exercise 12.8

Determine the force in members AG and CE of the truss shown. State whether the members are in tension or compression.



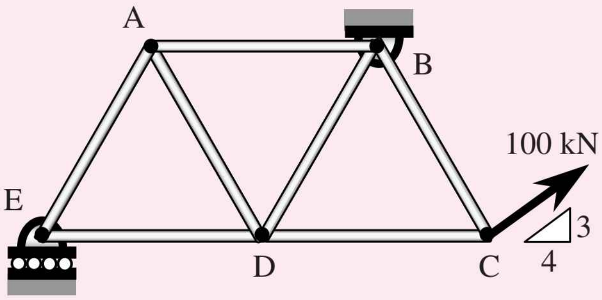
Exercise 12.9

A simple trust is loaded as shown. Determine the forces in members AB, BH GH and CF and state whether they are in tension or compression.



Exercise 12.10

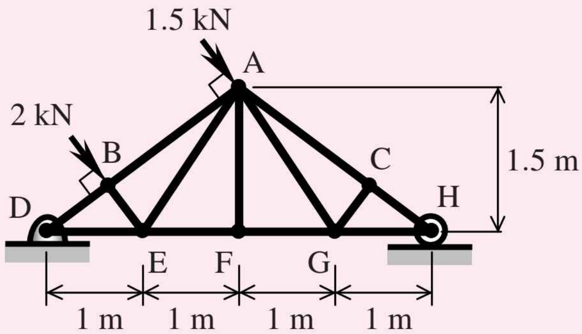
The truss shown is built from seven 2 m length members. Determine the force in members AB, AD and CD. State whether the members are in tension or compression.



Exercise 12.11

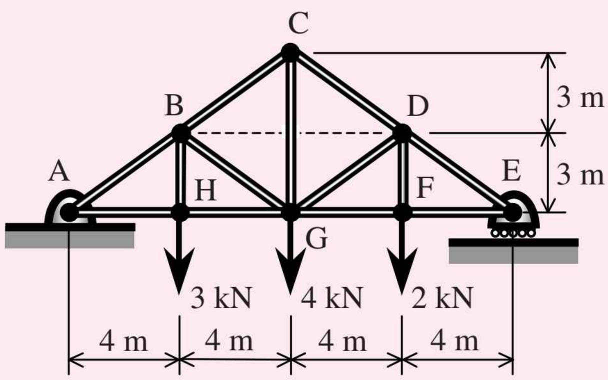
A trust is loaded as shown. Determine the forces in members AC, AG and FG and state whether they are in tension or compression using

a. method of joints,
b. method of sections.



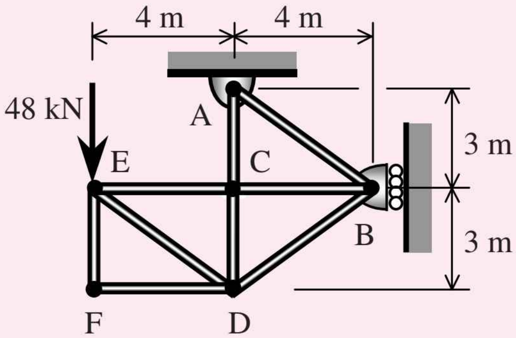
Exercise 12.12

Determine the force in members BC, BG and HG of the truss shown. State whether the members are in tension or compression.



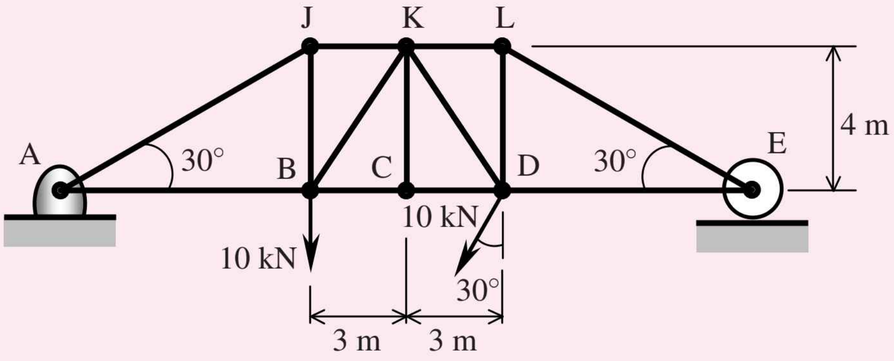
Exercise 12.13

Determine the force in all members of the truss shown. State whether the members are in tension or compression.



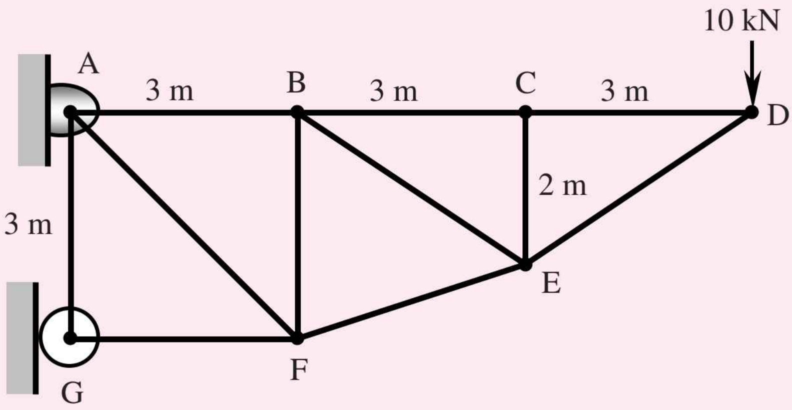
Exercise 12.14

Determine the force in members JK, BJ and AB and state whether the members are in tension or compression.



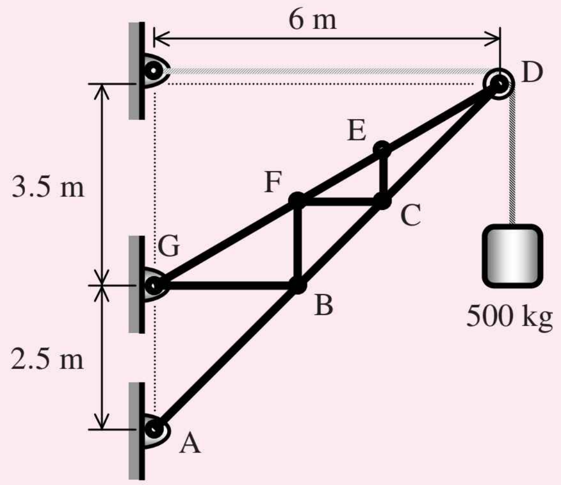
Exercise 12.15

Determine the force in members BC, BE and EF of the truss shown. State whether the members are in tension or compression.



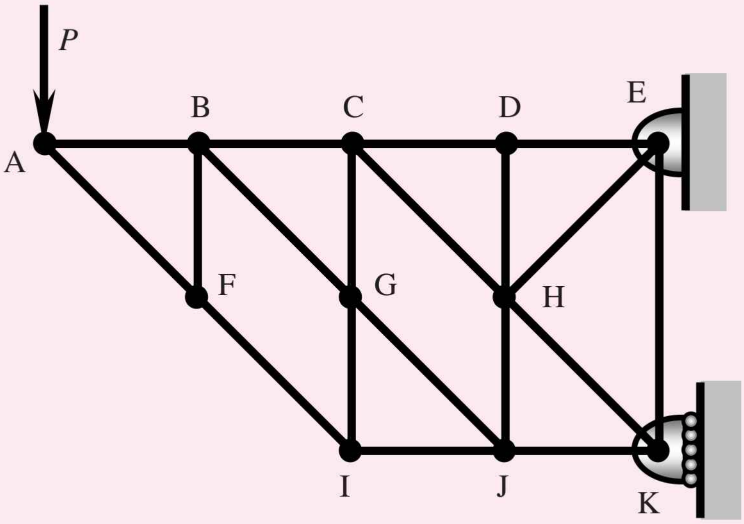
Exercise 12.16

The truss in the figure is used to support the 500 kg load at the pulley D. Determine the forces in all members if the radius of the pulley is 0.1 m. Members BG and CF are horizontal while members BF and CE are vertical. State also whether the members are in tension or compression.



Exercise 12.17

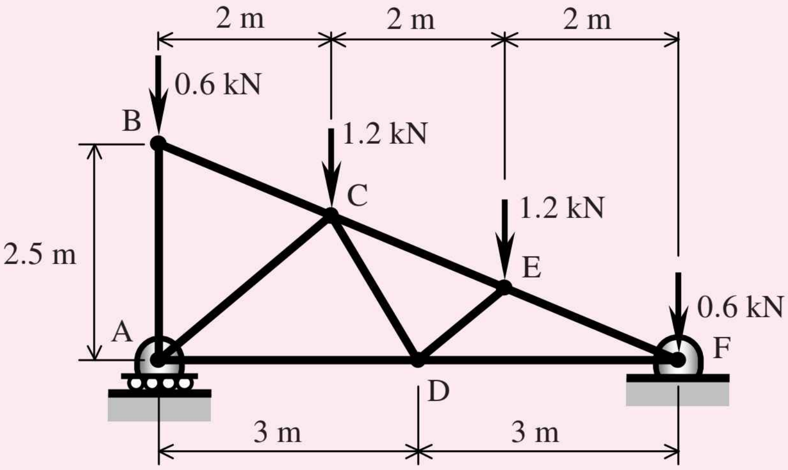
The figure shows a truss with a vertical load P at A and supported by pin E and roller K. Determine whether each member is in tension (T), compression (C) or a zero-force member (O) analytically, without calculations.



Answer: AB(T), BC(T), CD(T), DE(T), AF(C), FI(C), BG(O), GJ(O), CH(C), HK(C), HE(O), BF(O), CG(T), GI(T), DH(O), HJ(O), EK(T), IJ(C), JK(C)

Exercise 12.18

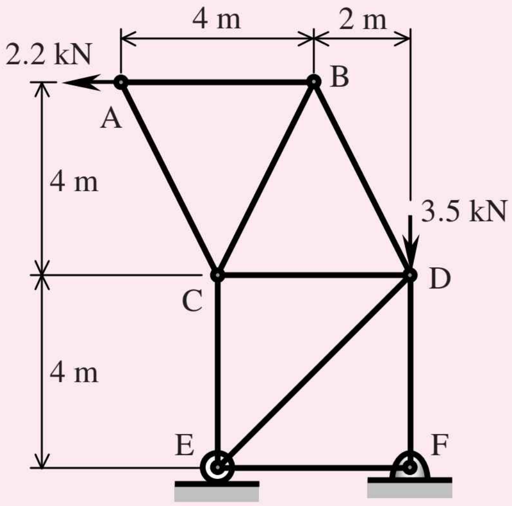
Determine the force in members AC, CD and DE, and state whether the members are in tension or compression.



Exercise 12.19

A trust is loaded as shown. Determine the forces in members DE, DF and CE and state whether they are in tension or compression using

a. method of joints,
b. method of sections.

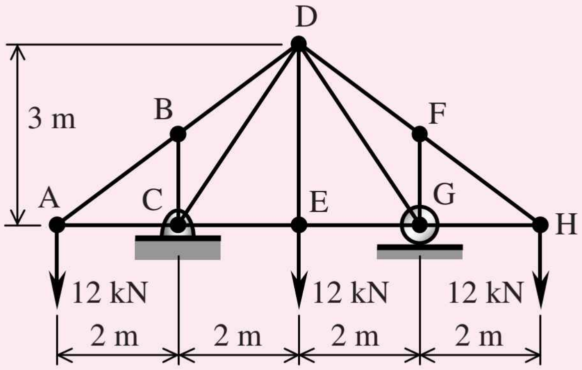


Exercise 12.20

Determine the force in members DF and DG of the truss shown by

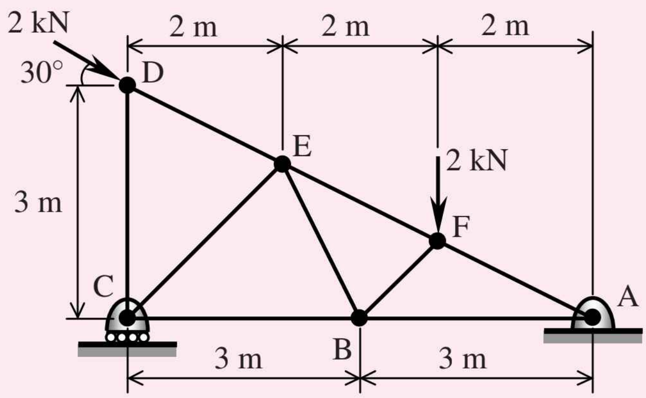
a. method of joints,
b. method of sections.

State whether the members are in tension (T) or compression (C).



Exercise 12.21

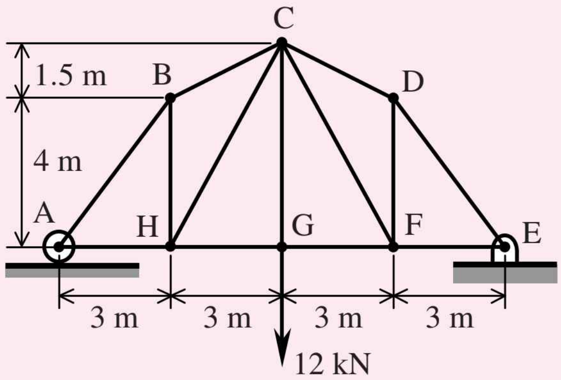
Determine the force in members BC, BE and EF, and state whether the members are in tension (T) or compression (C).



Exercise 12.22

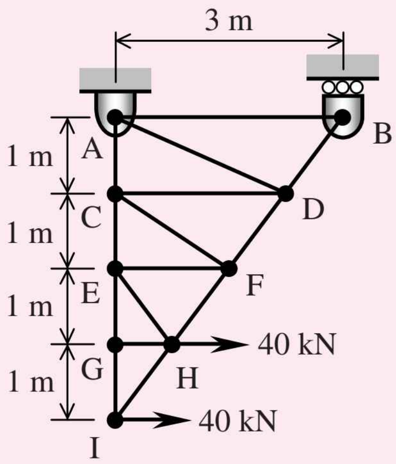
a. Determine the force in members BC and BH using the method of joints.
b. Determine the force in member GH using the method of section.

State whether the members are in tension (T) or compression (C).



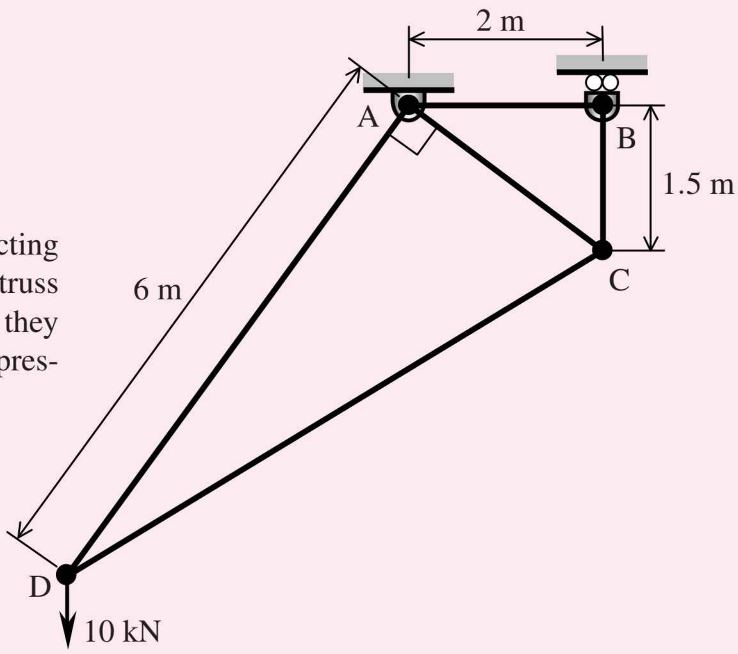
Exercise 12.23

a. Determine the force acting on members EF, CE, CF and DF of the truss shown and state whether they are in tension (T) or compression (C).
b. Determine the force acting on members AC, AD and BD of the truss shown and state whether they are in tension (T) or compression (C).



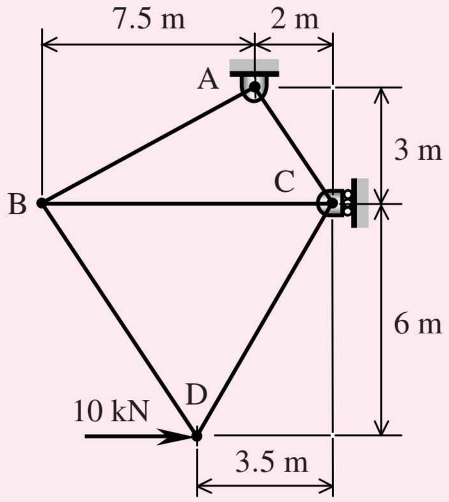
Exercise 12.24

Determine the force acting on all members of the truss shown and state whether they are in tension (T) or compression (C).



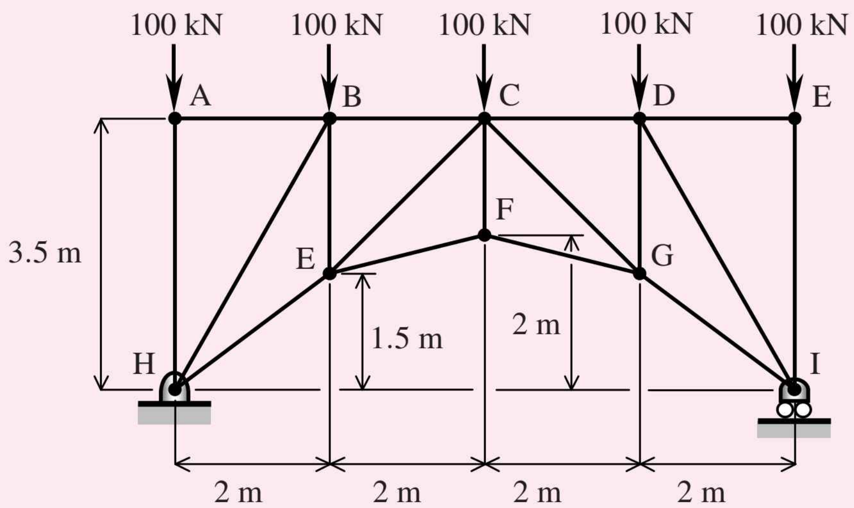
Exercise 12.25

Determine the force acting on all members of the truss shown and state whether they are in tension (T) or compression (C).



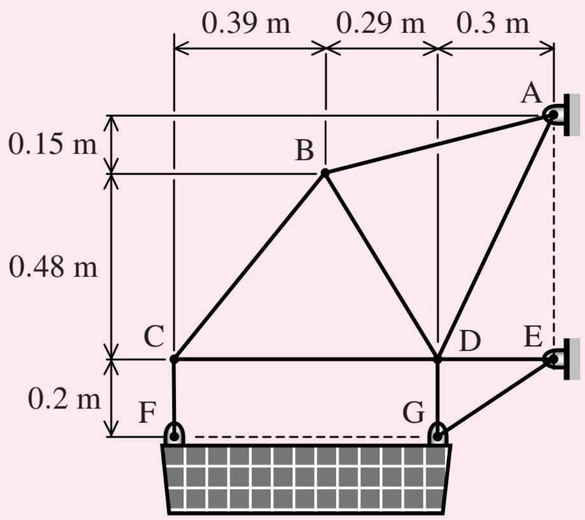
Exercise 12.26

a. Determine the force acting on members CD, CG and FG of the truss shown and state whether they are in tension (T) or compression (C).
b. Determine the force acting on members AB, BH and EH of the truss shown and state whether they are in tension (T) or compression (C).



Exercise 12.27

The truss shown is supporting a uniform load of 10 kN. Determine the force acting on members AD, BC, and BD and state whether they are in tension (T) or compression (C).



12.4 Analysis of frames and machines

Frames and machines are defined as rigid bodies comprising of at least one multiforce member (member with at least three forces acting). Frames are usually designed for supporting loads and are usually stationary, while machines are designed to modify and transmit forces, which are not necessarily stationary and always contain moving parts. Nevertheless, a correct FBD is very important in the initial step of solving problems. A similar strategy is used for the analysis of both frames and machines problems.

Two common types of problems include:

- To determine the components of forces acting on a member
- To determine the forces/ reactions at selected locations

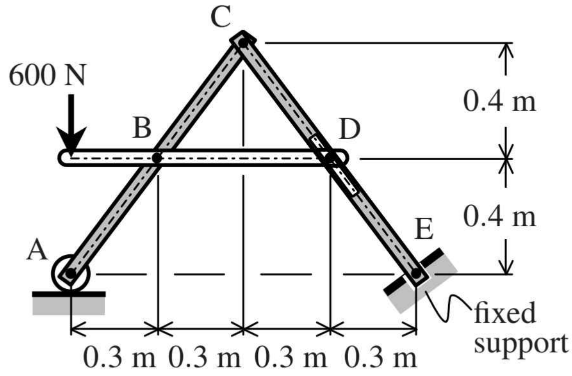
In the analysis of frames and machines, it is usually sufficient to determine the components of the force, unlike problems involving equilibrium of particles and rigid bodies where the resultant (magnitude and direction) of the force/ reaction is determined after calculating the components.

Strategy to determine the components of forces acting on a member:

- Separate the required member and draw the FBD. Identify all unknown forces/ components. Remember that for a two-force member, there is only one unknown. Since there are only three equations ($\sum F_x = 0$, $\sum F_y = 0$ and $\sum M_A = 0$ where A is any point) to solve equilibrium problems, any extra unknowns must be settled from 'another location'.
- 'Another location' means:
  - try overall FBD
  - try solving a member where there is a known force (magnitude and direction)
- Calculations can then be done on the required member when there are left with only three unknown forces/components.

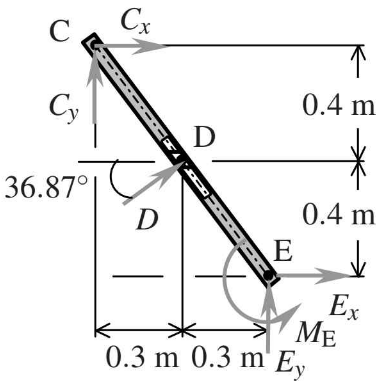
■ **Example 12.8**

Determine all force components acting on member CDE.



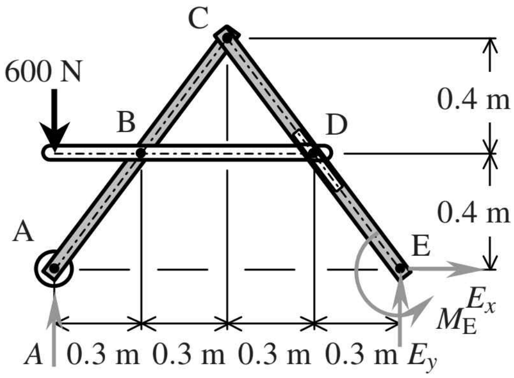
Solution:

1. Separate the required member and draw the FBD. Determine all force components.



There are 6 unknowns, namely, C_x, C_y, $\boldsymbol{D}$, E_x, E_y and M_E. Therefore 3 unknowns must be solved at 'another location'.

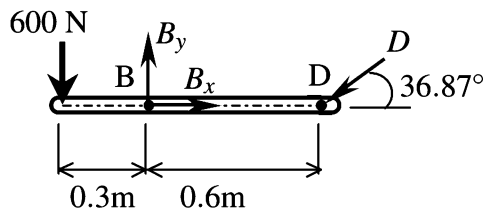
2. Try overall FBD.



The necessary equation is;

$$(+\rightarrow)\sum F_x = 0$$
$$\therefore E_x = 0$$

3. Try a member with a known force (at points B and D)



The necessary equation is;

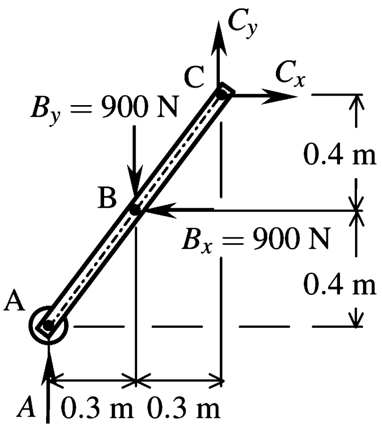
$$(+\circlearrowleft)\sum M_B = 0$$
$$D\sin 36.87°(0.6\text{ m}) - 600\text{ N}(0.3\text{ m}) = 0$$
$$\therefore D = 500\text{ N}$$

The (+ve) answer indicates that the assumed direction is correct, and observe that the reaction D on member BD is opposite in direction with the reaction D on member CDE. 2 unknowns have been solved, still one short of the required 3, hence continue analysis on member BD.

$$(+\rightarrow)\sum F_x = 0$$
$$B_x - D\cos 36.87° = 0$$
$$\therefore B_x = 400\text{ N}(\rightarrow)$$

$$(+\uparrow)\sum F_y = 0$$
$$B_y - 600 - D\sin 36.87° = 0$$
$$\therefore B_y = 900\text{ N}(\uparrow)$$

The (+ve) answers indicate that the assumed directions are correct.
4. Try member ABC, to solve either C_x or Cy.



Observe that the reaction components B_x and B_y are opposite in direction to B_x and B_y acting on member BD.

$$(+\rightarrow)\sum F_x = 0$$
$$C_x - 400\text{ N} = 0$$
$$\therefore C_x = 400\text{ N}(\rightarrow)$$

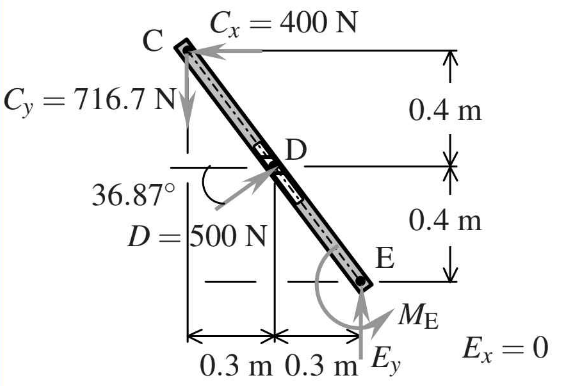
The (+ve) answer indicates that the assumed direction is correct.

Solving for moment at point A yields;

$$(+\circlearrowleft)\sum M_A = 0$$
$$400\text{ N}(0.4\text{ m}) + C_y(0.6\text{ m}) - 900\text{ N}(0.3\text{ m}) - 400\text{ N}(0.8\text{ m}) = 0$$
$$\therefore C_y = 716.7\text{ N}(\uparrow)$$

The (+ve) answer indicates that the assumed direction is correct.

5. Look back to the required member, CDE.



Observe that the components D, C_x and C_y are opposite in directions with reaction D on member BD and components C_x and C_y on member ABC.

$$(+\uparrow)\sum F_y = 0$$
$$-716.7\text{ N} + 500\sin 36.87^\circ + E_y = 0$$
$$\therefore E_y = 416.7\text{ N}(\uparrow)$$

The (+ve) answer indicates that the assumed direction is correct.

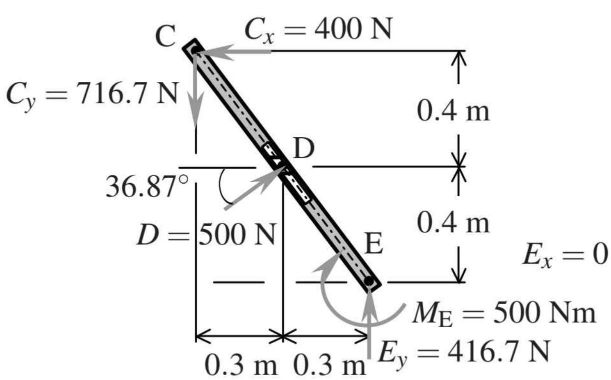
Solving for moment at point C yields;

$$(+\circlearrowleft)\sum M_C = 0$$
$$500\text{ N}(0.5\text{ m}) + 416.7\text{ N}(0.6\text{ m}) + M_E = 0$$
$$M_E = -500\text{ Nm}$$
$$\therefore M_E = 500\text{ Nm}(\circlearrowright)$$

Checking by taking moment at point E yields;

$$(+\circlearrowleft)\sum M_E = 0$$
$$400\text{ N}(0.8\text{ m}) + 716.7\text{ N}(0.6\text{ m}) - 500\text{ N}(0.5\text{ m}) + M_E = 0$$
$$M_E = -500\text{ Nm}$$
$$\therefore M_E = 500\text{ Nm}(\circlearrowright)$$

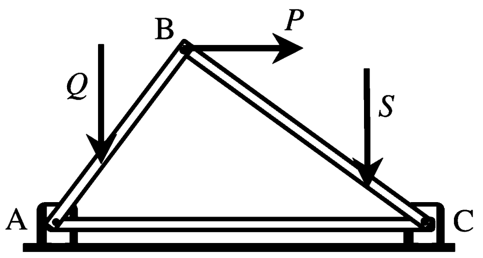
Therefore the complete answer is;

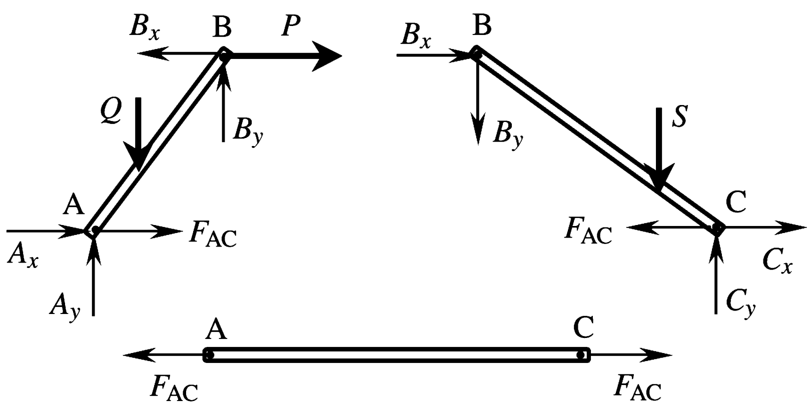


Observation:
- Indicate the chosen member when drawing FBD.
- The member used in calculations must be clearly stated.
- Notice that all assumptions in this question have been coincidently correct, any wrong assumption will produce a longer and tedious solution.

12.4.1 Special cases

- The need to solve simultaneous equations.
  If the discussed strategy is still unable to solve the problem, try to obtain two equations that can solve two unknown components simultaneously.
- Problems where a force is acting on a joint. Two possibilities:
  a. At least one member connected to the joint is a multiforce member. When the members are separated, the force is placed at one of the members only. For example;



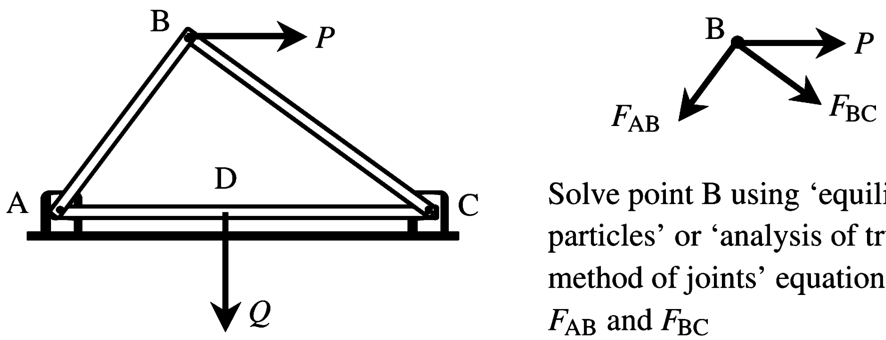


Observations:

∗ In the example above, force P can either be placed on member AB or member BC.

∗ The reactions at A and C are placed on members AB and BC as F_{AC}, to maintain AC as a two-force member.

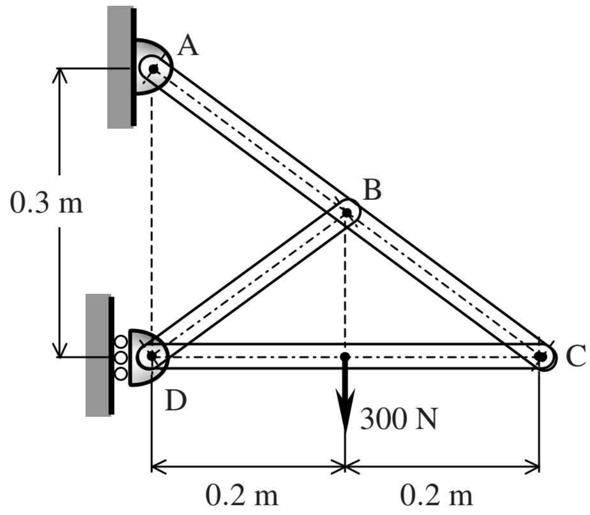
  b. All members connected to the joint are two-force members. Solve the joint using ‘equilibrium of particles’ or ‘analysis of trusses by method of joints’ equations. For example;



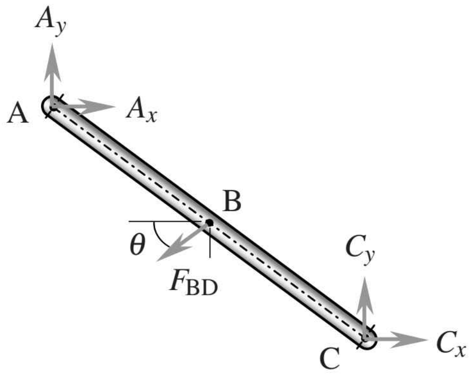
Solve point B using ‘equilibrium of particles’ or ‘analysis of trusses by method of joints’ equations to get F_{AB} and F_{BC}

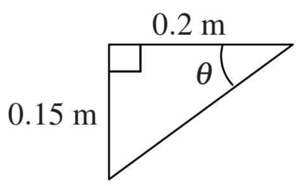
■ **Example 12.9**

Determine all force components acting on member ABC.



Solution:
Member ABC

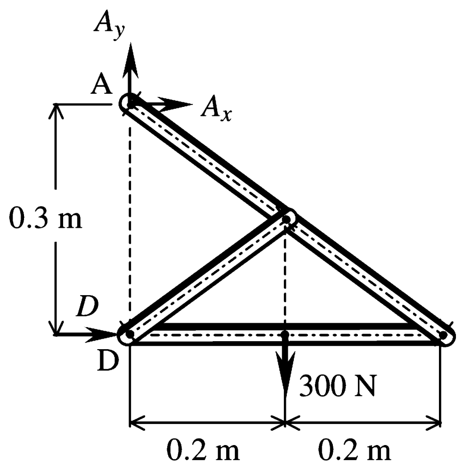




$$\theta = \tan^{-1}\left(\frac{0.15}{0.2}\right) = 36.87^\circ$$

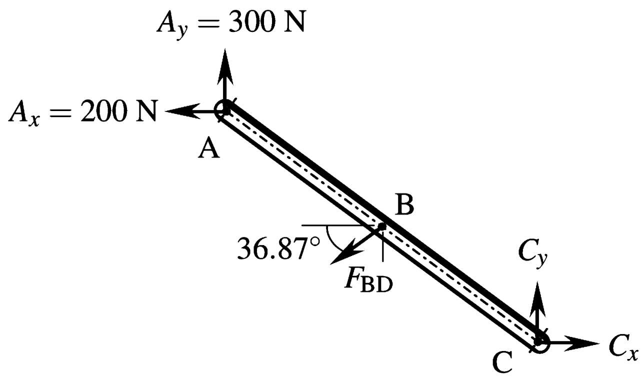
There are 5 unknowns, A_x, A_y, F_{BD}, C_x and C_y. The target is to determine 2 unknowns from another location.

Overall FBD



$$(+\circlearrowright)\sum M_A = 0$$
$$300(0.2) - D(0.3) = 0$$
$$\therefore D = 200\ \text{N}(\rightarrow)$$
$$(+\rightarrow)\sum F_x = 0$$
$$D + A_x = 0$$
$$A_x = -200\ \text{N}$$
$$\therefore A_x = 200\ \text{N}(\leftarrow)$$
$$(+\uparrow)\sum F_y = 0$$
$$A_y - 300 = 0$$
$$\therefore A_y = 300\ \text{N}(\uparrow)$$

Back to member ABC;

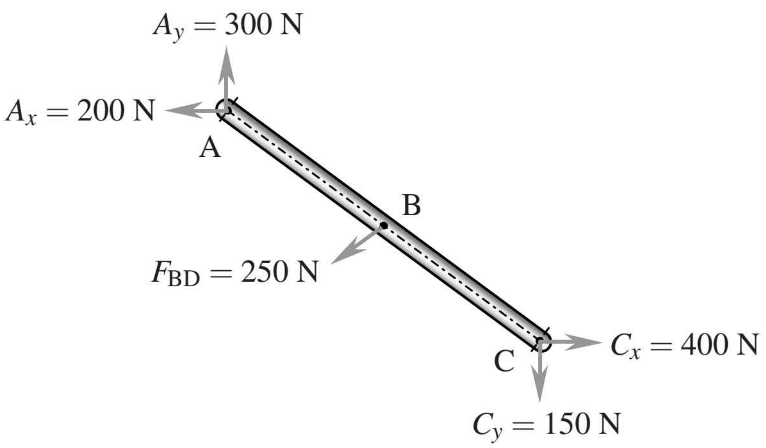


$$(+\circlearrowright)\sum M_C = 0$$
$$300(0.4) - 200(0.3) - F_{BD}\cos(36.87^\circ)(0.15) - F_{BD}\sin(36.87^\circ)(0.2) = 0$$
$$120 - 60 - 0.12F_{BD} - 0.12F_{BD} = 0$$
$$\therefore F_{BD} = 250\ \text{N}$$

$$(+\rightarrow)\sum F_x = 0$$
$$C_x - 200 - F_{BD}\cos(36.87^\circ) = 0$$
$$C_x - 200 - (250)\cos(36.87^\circ) = 0$$
$$\therefore C_x = 400\ \text{N}(\rightarrow)$$

$$(+\uparrow)\sum F_y = 0$$
$$C_y + 300 - F_{BD}\sin(36.87^\circ) = 0$$
$$C_y + 300 - (250)\sin(36.87^\circ) = 0$$
$$C_y = -150\ \text{N}$$
$$\therefore C_y = 150\ \text{N}(\downarrow)$$

For checking take the moment at B;



$$(+\circlearrowright)\sum M_B = 0$$

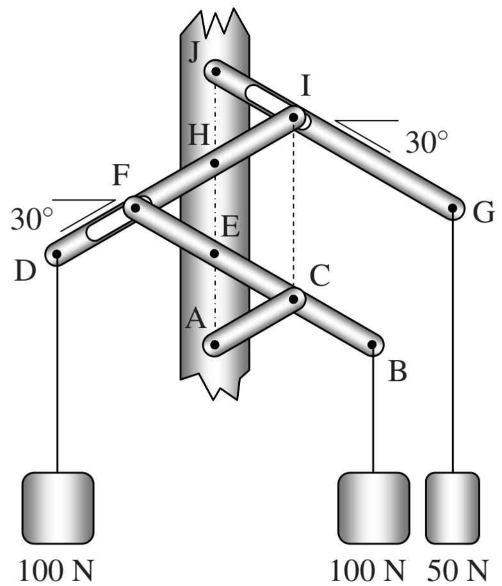
$$300(0.2) - 200(0.15) + 150(0.2) - 400(0.15) = 0$$

$$60 - 30 + 30 - 60 = 0 \ldots \text{checked!}$$

■ Example 12.10

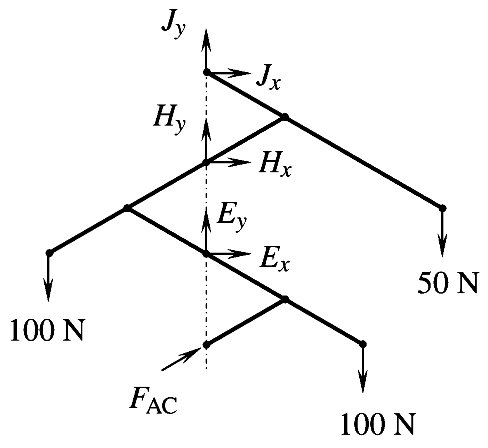
Determine the forces acting on member AC and state whether it is in tension (T) or compression (C). AC is parallel to DFHI and BCEF is parallel to GIJ.

Given:
AC = BC = CE = EF = 0.5 m
DF = FH = HI = IJ = 0.5 m
IG = 1 m



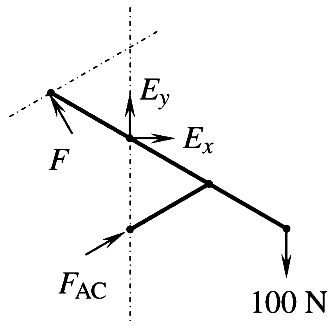
Solution:

Try using overall FBD



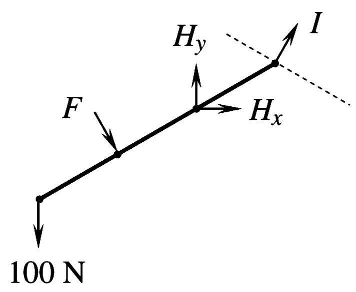
There are 7 unknowns, F_{AC}, E_x, E_y, H_x, H_y, J_x and J_y. Do not start here! Too many unknowns.

Try using BCEF



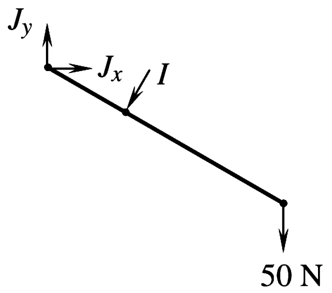
Try a member where there is a known force. There are 4 unknowns for member BCEF; F_{AC}, E_x, E_y and F. Do not start here! Too many unknowns.

Try using DFHI



There are 4 unknowns for member DFHI; H_x, H_y, I and F. Do not start here! Too many unknowns.

Try using GIJ

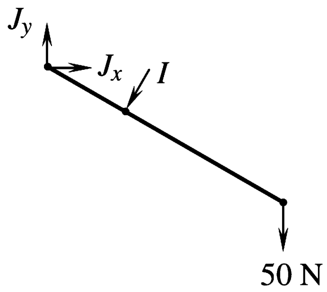


There are 3 unknowns for member DFHI; J_x, J_y, and I. Start here! Number of unknowns is the same as the number of equations available.

The solution strategy:

- FBD of member GIJ $\Rightarrow \sum M_J = 0$, solve for I
- FBD of member DFHI $\Rightarrow \sum M_H = 0$, solve for F since I is now known.
- FBD of member BCEF $\Rightarrow \sum M_E = 0$, solve for F_{AC} since F is now known.

FBD of GIJ



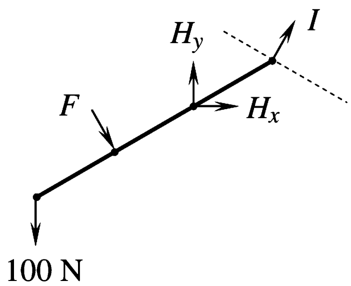
$$(+\circlearrowright)\sum M_J = 0$$

$$I(0.5) + 50(1.5\cos 30^\circ) = 0$$

$$I = -\frac{50(1.5\cos 30^\circ)}{0.5} = -130\text{ N}$$

$$\therefore I = 130\text{ N}\measuredangle 60^\circ$$

FBD of DFHI



$$(+\circlearrowright)\sum M_H = 0$$

$$I\sin 60^\circ(0.5\cos 30^\circ) - I\sin 60^\circ(0.5\sin 30^\circ)$$
$$-F(0.5) - 100(1.0\cos 30^\circ) = 0$$

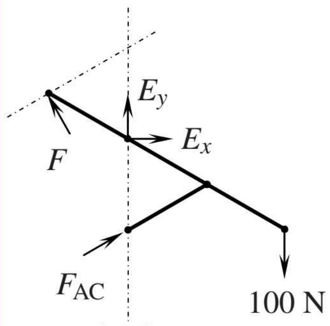
$$130\sin 60^\circ(0.5\cos 30^\circ) - 130\sin 60^\circ(0.5\sin 30^\circ)$$
$$-F(0.5) - 100(1.0\cos 30^\circ) = 0$$

$$48.75 - 28.15 - 0.5F - 86.6 = 0$$

$$F = -66\text{ N}$$

$$\therefore F = 66\text{ N}\measuredangle 60^\circ$$

FBD of BCEF



$$(+\circlearrowright)\sum M_{\rm E} = 0$$

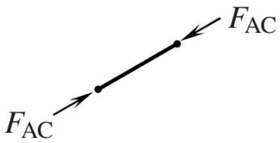
$$-F\sin 30^\circ(0.5) - F_{\rm AC}\cos 30^\circ)(0.5) + 100(1.0\cos 30^\circ) = 0$$

$$-66\sin 30^\circ(0.5) - F_{\rm AC}\cos 30^\circ(0.5) + 100(1.0\cos 30^\circ) = 0$$

$$-16.5 - 0.433F_{\rm AC} + 86.6 = 0$$

$$\therefore F_{\rm AC} = 161.9\ \text{N}\measuredangle 30^\circ$$

FBD of AC



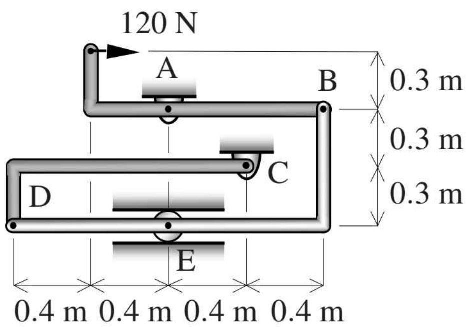
$\therefore$ member AC is in compression (C)

Observation

- Ensure that the FBD is correct.
- All directions (of reactions) are assumed.
- Notice that, wherever possible, the assumption for direction is assumed 'safely', to avoid confusion in determining the correct direction.
- Newton's Third Law is adhered to all the way. For example, the actual direction of force I is solved while solving member DHFI. Force I is then transferred to member GIJ, in the opposite direction.
- Just calculate the necessary parts required for solution. For example, there is no need to calculate E_x, E_y, H_x, H_y, J_x and J_y as this will consume time and the risks of making mistakes will be higher.

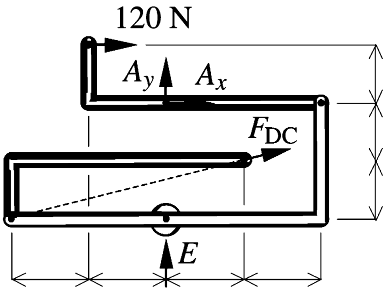
■ Example 12.11

The frame shown comprises of three members. Determine all components of forces acting on member DEB.



Solution:

From overall FBD;



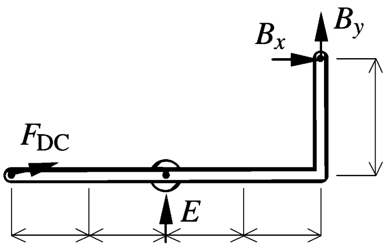
$(+\circlearrowleft) M_A = 0$

$120(0.3) - F_{DC}\cos 14.04°(0.3)$

$- F_{DC}\sin 14.04°(0.4) = 0$

$F_{DC} = 92.8$ N(positive, ↗)

From FBD of member DEB;



$(+\circlearrowleft) M_B = 0$

$F_{DC}\sin 14.04°(1.6) + 0.8E$

$- F_{DC}\cos 14.04°(0.6) = 0$

$(92.8)\sin 14.04°(1.6) + 0.8E$

$- (92.8)\cos 14.04°(0.6) = 0$

$E = 22.5$ N(positive, ↑)

$(+\rightarrow) F_x = 0$

$F_{DC}\cos 14.04° + B_x = 0$

$(92.8)\cos 14.04° + B_x = 0$

$B_x = -90$ N(negative, ←)

$(+\uparrow) F_y = 0$

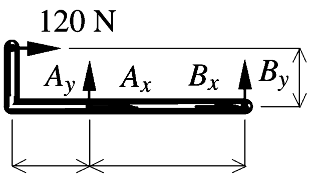
$F_{DC}\sin 14.04° + E + B_y = 0$

$(92.8)\sin 14.04° + (22.5) + B_y = 0$

$B_y = -45$ N(negative, ↓)

Or

From FBD of member AB;



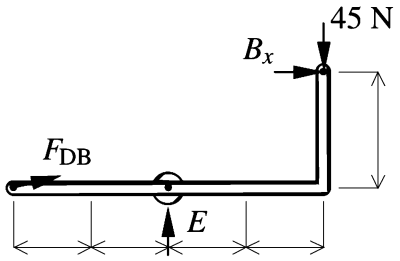
$(+\circlearrowleft) M_B = 0$

$120(0.3) - (0.8)B_y$

$B_y = 45$ N(positive, ↑)

From FBD of member DEB;

$$(+\rightarrow)F_x = 0$$
$$F_{DC}\cos 14.04^\circ + B_x = 0 \quad \ldots(1)$$
$$(+\uparrow)F_y = 0$$
$$F_{DC}\sin 14.04^\circ + E - B_y = 0$$
$$F_{DC}\sin 14.04^\circ + E - 45 = 0$$
$$0.242F_{DC} + E = 45 \quad \ldots(2)$$
$$(+\circlearrowright)M_B = 0$$
$$F_{DC}\sin 14.04^\circ(1.6) + 0.8E$$
$$- F_{DC}\cos 14.04^\circ(0.6) = 0$$
$$-0.194F_{DC} + 0.8E = 0 \quad \ldots(3)$$



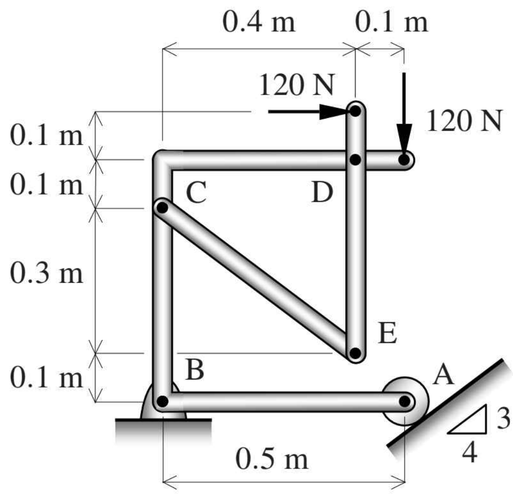
Solving simultaneous equations (2) and (3) gives;

$$E = 22.5\ \text{N(positive, }\uparrow)$$
$$F_{DC} = 92.8\ \text{N(positive, }\nearrow)$$

Substituting F_{DC} into equation (1) gives $B_x = -90$ N(negative,←)

▪ Example 12.12

The frame shown comprises of three members. Determine all components of forces acting on member ABCD.



Solution:

From overall FBD;

$$(+\circlearrowright)M_B = 0$$

$$120(0.6) + 120(0.5) - A(\frac{4}{5})(0.5) = 0$$

$$A = 330\ \text{N(positive, } 53.13^\circ \nwarrow)$$

$$(+\rightarrow)F_x = 0$$

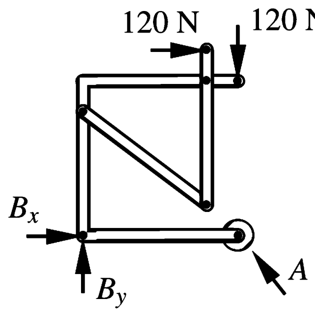
$$B_x + 120 - \frac{3}{5}(330) = 0$$

$$B_x = 78\ \text{N(positive, } \rightarrow)$$

$$(+\uparrow)F_y = 0$$

$$B_y - 120 + \frac{4}{5}(330) = 0$$

$$B_y = -144\ \text{N(negative, } \downarrow)$$



From FBD of member ABCD;

$$(+\circlearrowright)M_D = 0$$

$$120(0.1) + (\frac{3}{5})F_{CE}(0.4) + (\frac{4}{5})F_{CE}(0.1)$$
$$- 78(0.5) - 144(0.4) - (\frac{4}{5})(330)(0.1)$$
$$+ (\frac{3}{5})(330)(0.5) = 0$$

$$F_{CE} = 37.5\ \text{N(positive, } 36.87^\circ \nwarrow)$$

$$(+\rightarrow)F_x = 0$$

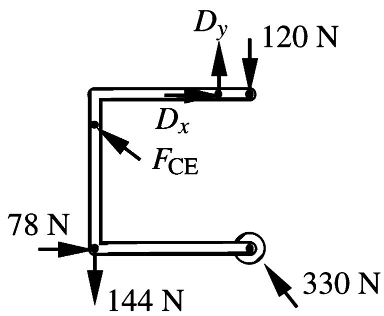
$$D_x - \frac{4}{5}(37.5) + 78 - \frac{3}{5}(330) = 0$$

$$D_x = 150\ \text{N(positive, } \rightarrow)$$

$$(+\uparrow)F_y = 0$$

$$D_y - 120 + \frac{3}{5}(37.5) - 144 + \frac{4}{5}(330) = 0$$

$$D_y = -22.5\ \text{N(negative, } \downarrow)$$



Or

From FBD of member DE;

$$(+\circlearrowright)M_D = 0$$

$$120(0.1) - (\frac{4}{5})F_{CE}(0.4) = 0$$

$$F_{CE} = 37.5\ \text{N(positive, } 36.87^\circ \searrow)$$

$$(+\rightarrow)F_x = 0$$

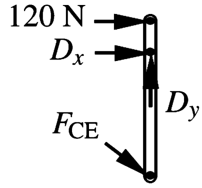
$$D_x + 120 + \frac{4}{5}(37.5) = 0$$

$$D_x = -150\ \text{N(negative, } \leftarrow)$$

$$(+\uparrow)F_y = 0$$

$$D_y - \frac{3}{5}(37.5) = 0$$

$$D_y = 22.5\ \text{N(positive, } \uparrow)$$



Therefore on member ABCD;

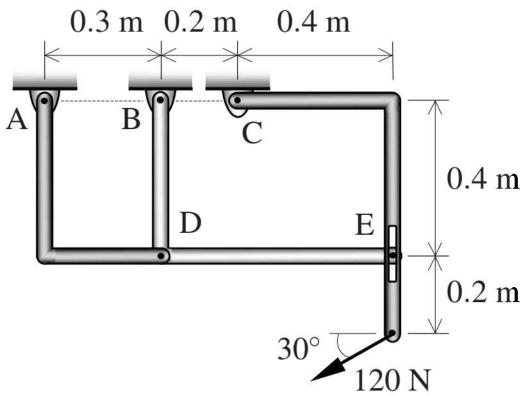
$$F_{CE} = 37.5\ \text{N}(36.87^\circ \nwarrow)$$

$$D_x = 150\ \text{N}(\rightarrow)$$

$$D_y = 22.5\ \text{N}(\downarrow)$$

■ Example 12.13

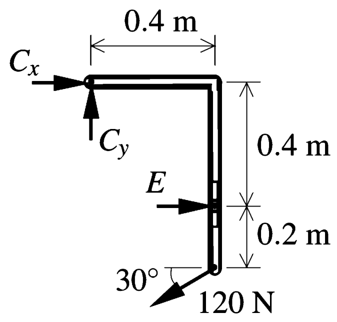
The frame shown comprises of three members. Determine all components of forces acting on member BDE.



Solution:

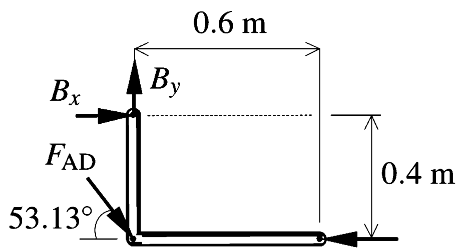
From FBD of member CE;

$$(+\circlearrowright)M_C = 0$$
$$120\cos 30°(0.6) + 120\sin 30°(0.4) - E(0.4) = 0$$
$$E = 215.9 \text{ N(positive, } \rightarrow)$$
$$(+\rightarrow)F_x = 0$$
$$C_x + E - 120\cos 30° = 0$$
$$C_x + 215.9 - 120\cos 30° = 0$$
$$C_x = -112 \text{ N(negative, } \leftarrow)$$
$$(+\uparrow)F_y = 0$$
$$C_y - 120\sin 30° = 0$$
$$C_y = 60 \text{ N(positive, } \uparrow)$$

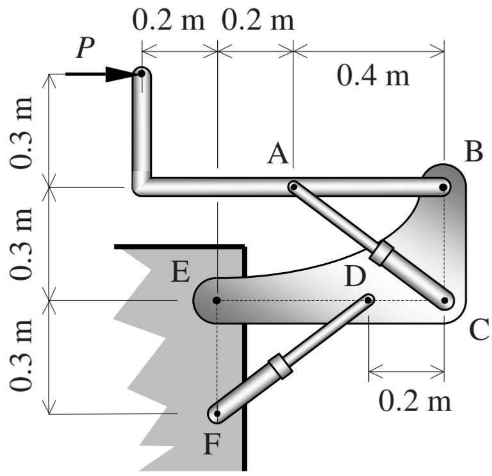


From FBD of member BDE;

$$(+\circlearrowright)M_B = 0$$
$$215.9(0.4) - F_{AD}\cos 53.13°(0.4) = 0$$
$$F_{AD} = 359.8 \text{ N(positive, } \searrow)$$
$$(+\rightarrow)F_x = 0$$
$$B_x - 215.9 + F_{AD}\cos 53.13° = 0$$
$$B_x - 215.9 + 359.8\cos 53.13° = 0$$
$$B_x = 0$$
$$(+\uparrow)F_y = 0$$
$$B_y - F_{AD}\sin 53.13° = 0$$
$$B_y - 359.8\sin 53.13° = 0$$
$$B_y = 287.8 \text{ N(positive, } \uparrow)$$



■ Example 12.14



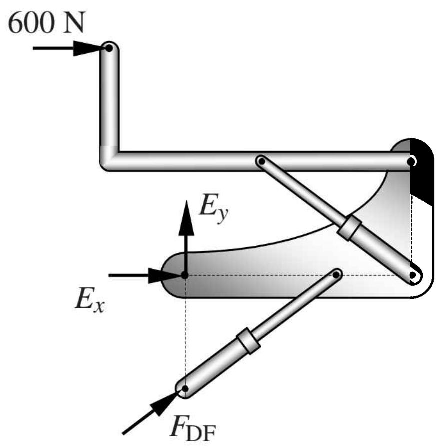
The mechanism shown is used to support the $P = 600$ N load.

(a) Determine the force in both hydraulic cylinders and state whether the hydraulic cylinders are in tension or compression.

(b) Determine also components of the reaction at pins B, C, D, and E acting on member BCDE.

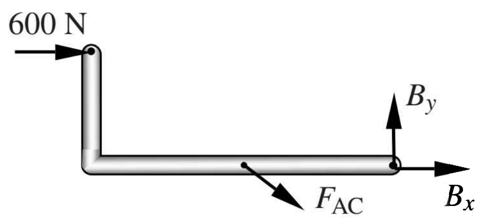
Solution:

From overall FBD;



$$(+\circlearrowright)M_E = 0$$
$$600(0.6) - F_{DF}\cos 36.87^\circ(0.3) = 0$$
$$F_{DF} = 1500 \text{ N(positive value, C)}$$
$$(+\rightarrow)F_x = 0$$
$$E_x + 600 + 1500\cos 36.87^\circ = 0$$
$$E_x = -1800 \text{ N(negative value, } \leftarrow)$$
$$(+\uparrow)F_y = 0$$
$$E_y + 1500\sin 36.87^\circ = 0$$
$$E_y = -900 \text{ N(negative value, } \downarrow)$$

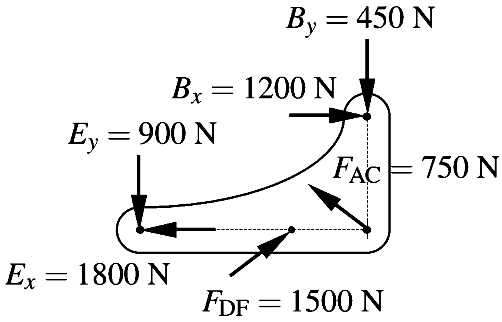
From FBD of member AB;



$$(+\circlearrowright)M_B = 0$$
$$600(0.3) - F_{AC}\sin 36.87^\circ(0.4) = 0$$
$$F_{AC} = 750 \text{ N(positive value, T)}$$
$$(+\rightarrow)F_x = 0$$
$$B_x + 600 + 750\cos 36.87^\circ = 0$$
$$B_x = -1200 \text{ N(negative value, } \leftarrow)$$
$$(+\uparrow)F_y = 0$$
$$B_y - 750\sin 36.87^\circ = 0$$
$$B_y = 450 \text{ N(positive value, } \uparrow)$$

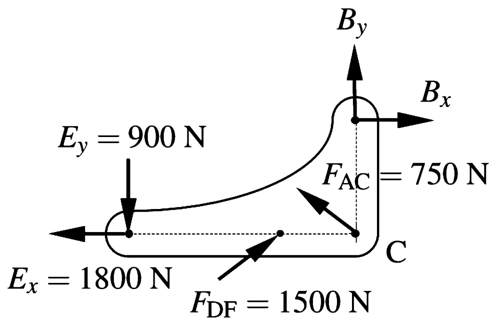
From FBD of member BCDE;

$F_{DF} = 1500$ N(↗) at pin D
$F_{AC} = 750$ N(↖) at pin C
$E_x = 1800$ N(←) at pin E
$E_y = 900$ N(↓) at pin E
$B_x = 1200$ N(→) at pin B
$B_y = 450$ N(↓) at pin B



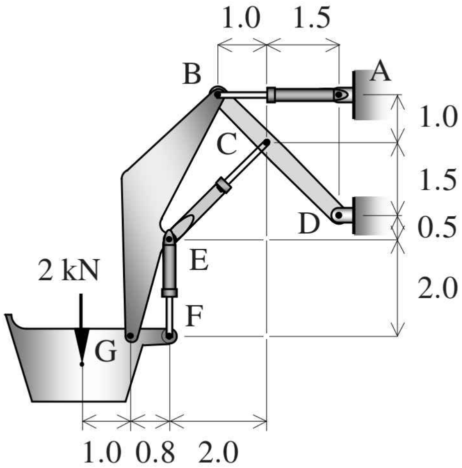
Or

From FBD of member BCDE;



$$(+\uparrow) F_y = 0$$
$$-900 + 1500\sin 36.87^\circ + 750\sin 36.87^\circ + B_y = 0$$
$$B_y = -450 \text{ N(negative value, } \downarrow)$$
$$(+\rightarrow) F_x = 0$$
$$-1800 + 1500\cos 36.87^\circ - 750\cos 36.87^\circ + B_x = 0$$
$$B_x = 1200 \text{ N(positive value, } \rightarrow)$$
$$(+\circlearrowright) M_C = 0$$
$$-(900)(0.6) + (1500\sin 36.87^\circ)(0.2) + (B_x)(0.3) = 0 \ldots \text{checked!}$$

■ Example 12.15

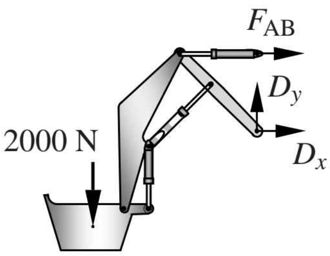


Determine the force in hydraulic cylinders AB, CE and EF for the mechanism shown to support the 2 kN load. State whether the cylinders are in tension (T) or compression (C). All dimensions are in meters.

Solution:

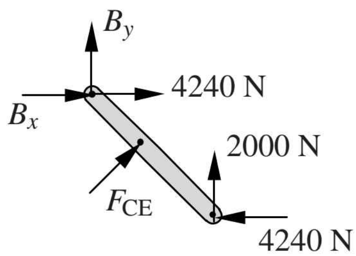
From overall FBD;

$$(+\circlearrowleft)M_{\mathrm{D}} = 0$$
$$2000(5.3) - F_{\mathrm{AB}}(2.5) = 0$$
$$F_{\mathrm{AB}} = 4240\ \mathrm{N(T)}$$
$$(+\rightarrow)F_x = 0$$
$$F_{\mathrm{AB}} + D_x = 0$$
$$D_x = -4240\ \mathrm{N}(\leftarrow)$$
$$(+\uparrow)F_y = 0$$
$$D_y - 2000 = 0$$
$$D_y = 2000\ \mathrm{N}(\uparrow)$$



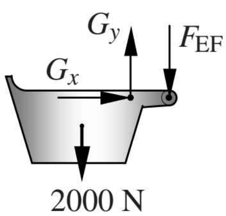
From FBD of member BCD;

$$(+\circlearrowleft)M_{\mathrm{B}} = 0$$
$$4240(2.5) - 2000(2.5) - F_{\mathrm{CE}}(\sqrt{2}) = 0$$
$$F_{\mathrm{CE}} = 3960\ \mathrm{N(C)}$$



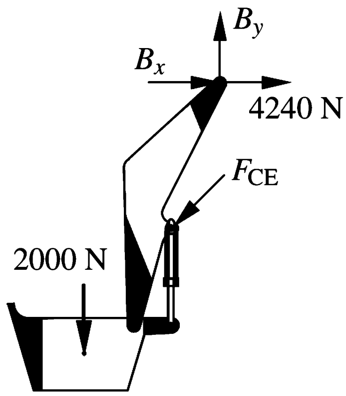
From FBD of member GF;

$$(+\circlearrowleft)M_{\mathrm{G}} = 0$$
$$2000(1.0) - F_{\mathrm{EF}}(0.8) = 0$$
$$F_{\mathrm{EF}} = 2500\ \mathrm{N(C)}$$



Check FBD of member BEFG;

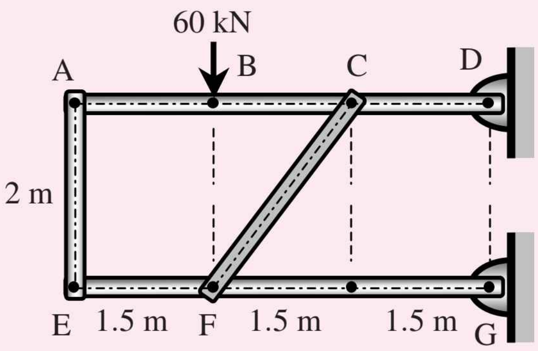
$$(+\circlearrowleft) M_B = 0$$
$$2000(2.8) + F_{CE}\cos 45°(1.0) - F_{CE}\sin 45°(3.0) = 0$$
$$5600 + 0.707F_{CE} - 2.121F_{CE} = 0$$
$$F_{CE} = 3960 \text{ N(C)} \ldots \text{checked!}$$



12.5 Example questions (frames)

Exercise 12.28

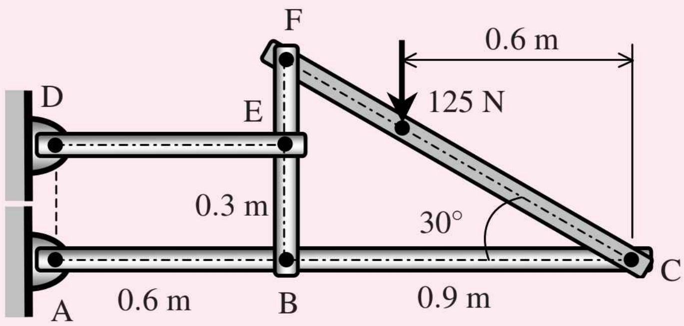
a. Determine all components of forces acting on member EFG.
b. Determine all components of forces acting on member ABCD.



Answer:
(a) $G_x = 90$ kN (←), $G_y = 40$ kN (↓), $F_{CF} = 150$ kN (∡53.13°) and $F_{AE} = 80$ kN (↓)
(b) $F_{EA} = 80$ kN (↑), $F_B = 60$ kN (↓), $F_{FC} = 150$ kN (↗53.13°), $D_x = 90$ kN (→) and $D_y = 100$ kN (↑)

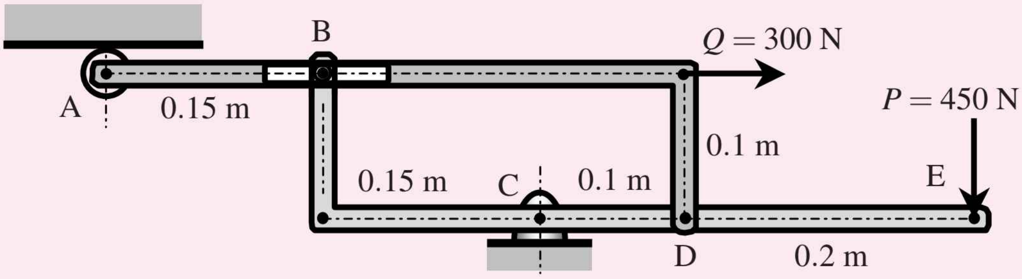
Exercise 12.29

a. Determine all components of forces acting on member ABC.
b. Determine all components of forces acting on member FEB.



Exercise 12.30

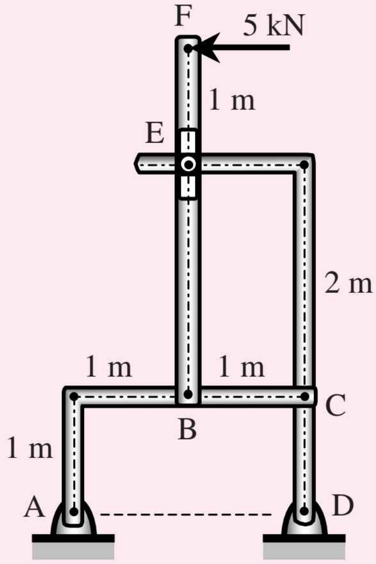
Determine all components of forces acting on member BDE.



Answer: $B_x = 760$ N(↓), $C_x = 300$ N(←), $C_y = 1000$ N(↑), $D_x = 300$ N(←), $D_y = 210$ N(↑) and $E = 450$ N(↓)

Exercise 12.31

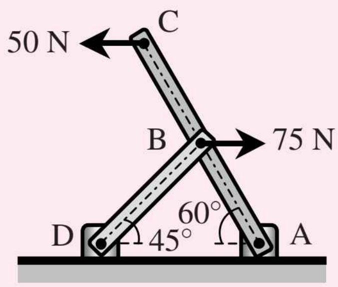
a. Determine all components of forces acting on member DCE.
b. Determine all components of forces acting on member ABC.
c. Determine all components of forces acting on member BEF.



Exercise 12.32

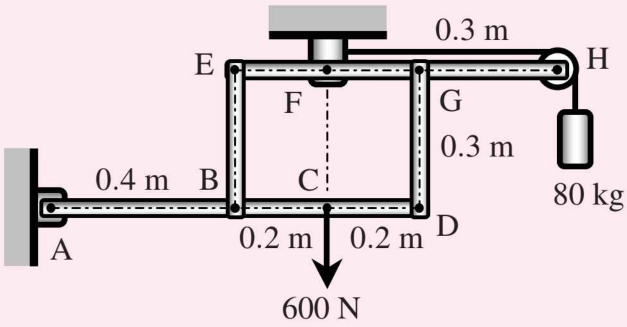
Determine components of the reactions at A and D.

Given AB = BC = 0.2 m.



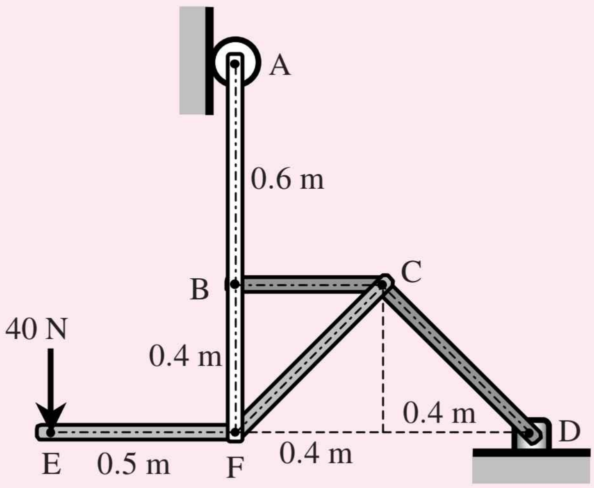
Exercise 12.33

a. Determine all components of forces acting on member ABCD.
b. Determine all components of forces acting on member EFGH.



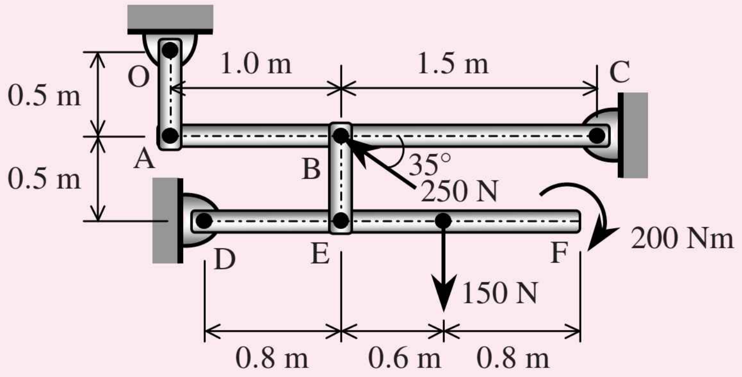
Exercise 12.34

a. Determine all components of forces acting on member CFE.
b. Determine all components of forces acting on member ABF.
c. Determine all components of forces acting on member BCD.



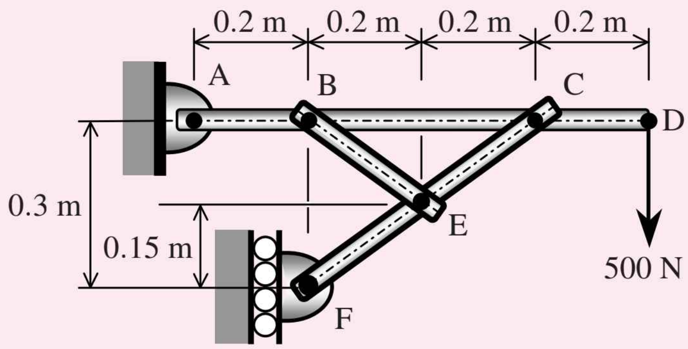
Exercise 12.35

Determine all components of forces acting on member ABC.



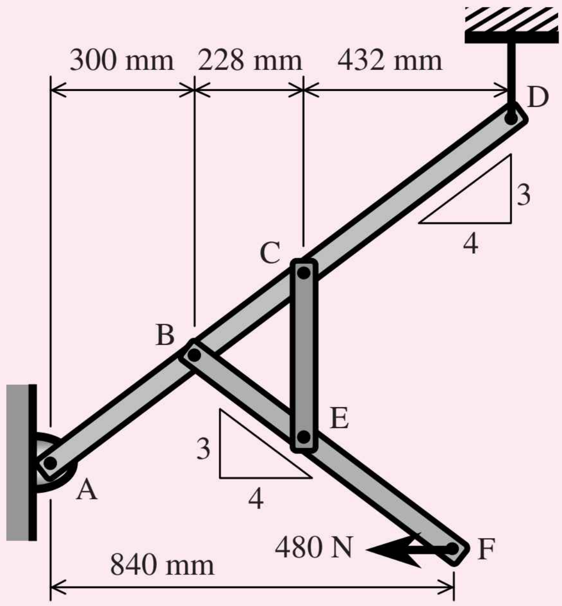
Exercise 12.36

Determine all components of forces acting on member ABCD.



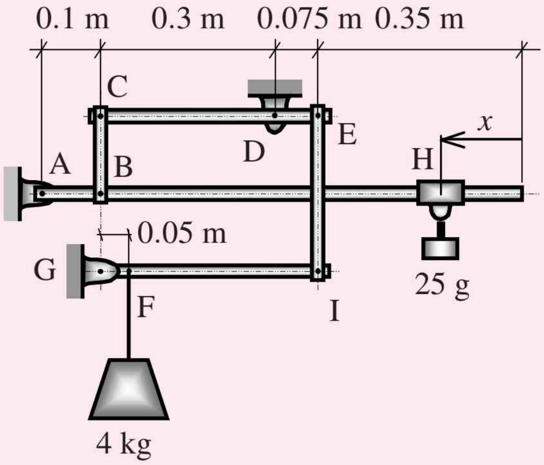
Exercise 12.37

Determine all components of forces acting on member ABCD.



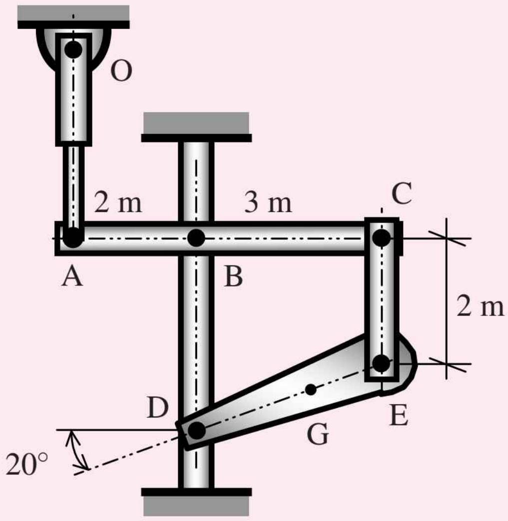
Exercise 12.38

The mechanism is used to maintain the 4 kg and 25 g masses at the position shown. Determine all components acting on member ABH and the distance x.



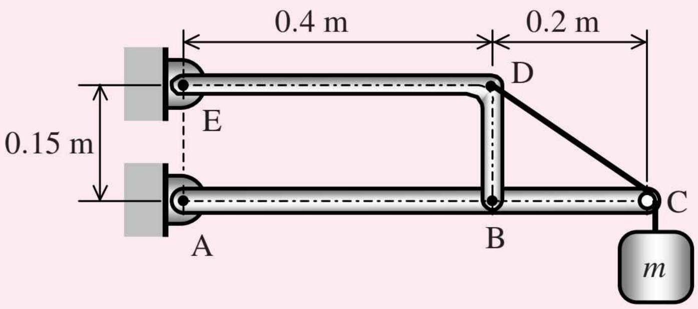
Exercise 12.39

Position of the 30 kg member DE is controlled by hydraulic cylinder OA. The centre of gravity of member DE is located at G with DG = 2.1 m. Determine the force in the hydraulic cylinder and all components on member ABC for the position shown.



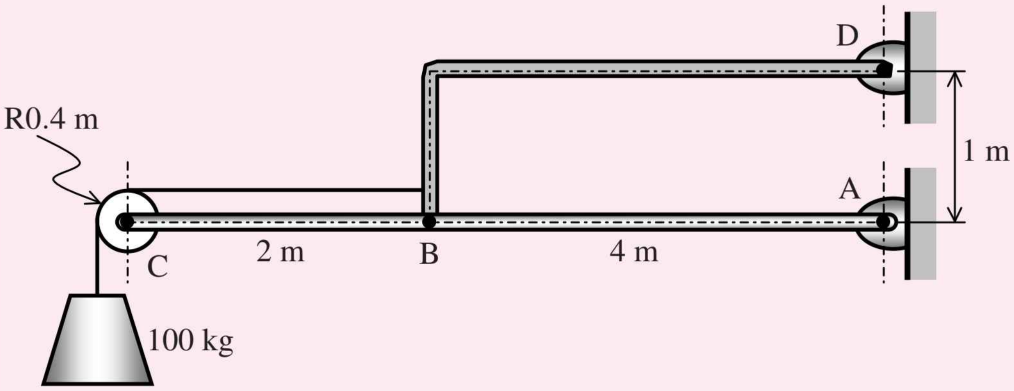
Exercise 12.40

Determine the horizontal and vertical components of the forces exerted on member ABC. Neglect the size of the small pulley at C. The mass $m = 100$ kg.



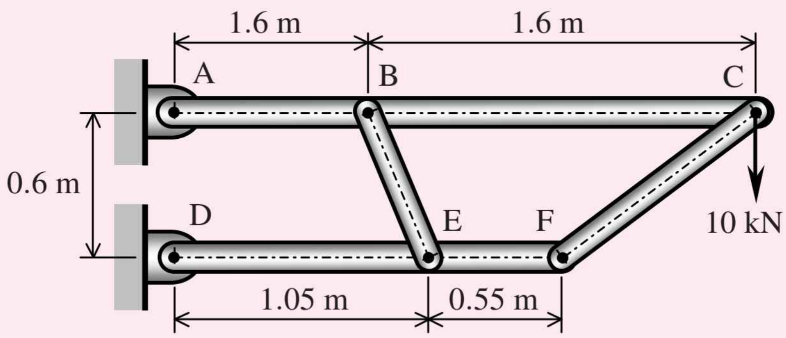
Exercise 12.41

Determine the horizontal and vertical components of the forces exerted on member ABC.



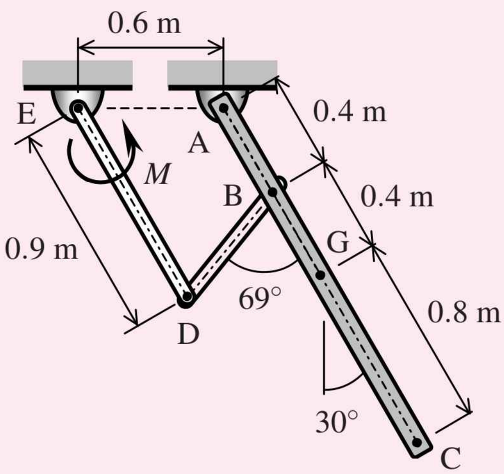
Exercise 12.42

a. Determine all components of forces acting on member ABC.
b. Determine all components of forces acting on member DEF.
c. Determine all components of reaction at pins A and D.

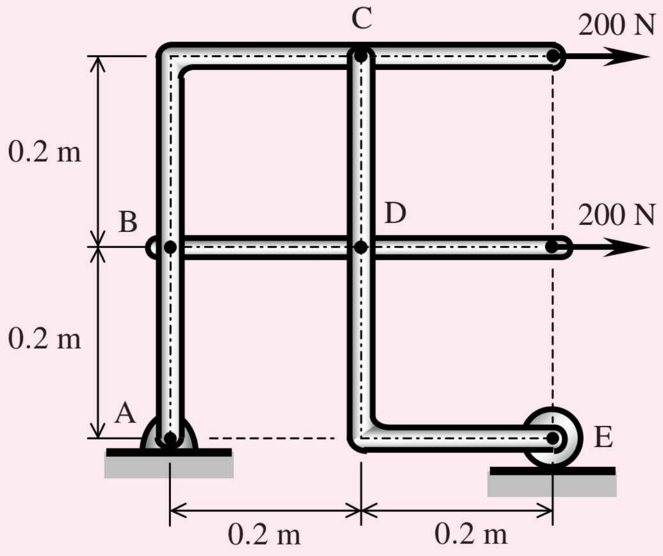


Exercise 12.43

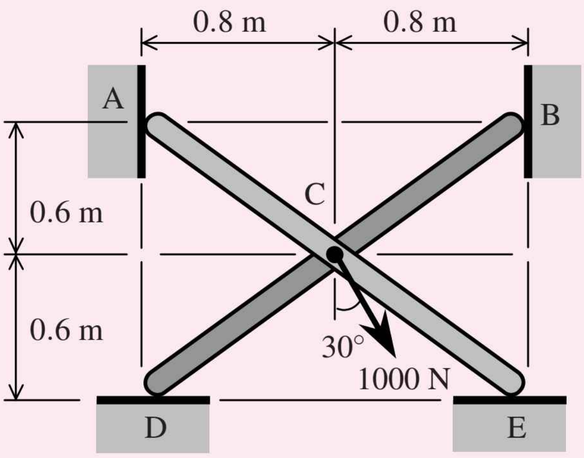
The 80 kg door AC with mass centre at G is held in the open position by means of a moment M applied at E. Member ED is parallel to the door for the 30° position shown in the figure. Determine moment M and the force in member BD. Neglect the mass of members BD and DE.



Exercise 12.44
Determine all force components acting on member ABC for the frame shown.

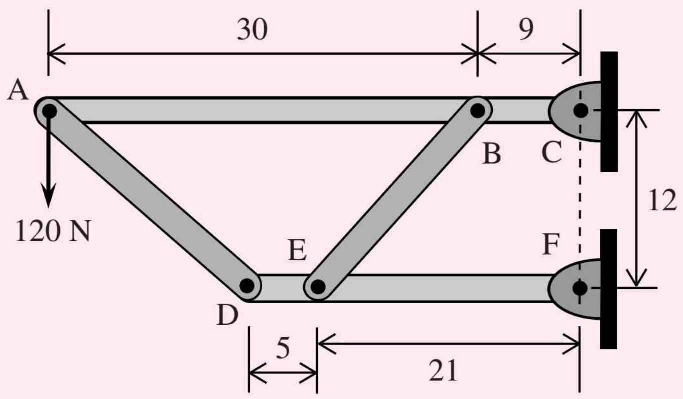


Exercise 12.45
Determine the reactions at A,B,D and E of the frame shown. All contacting surfaces are smooth.



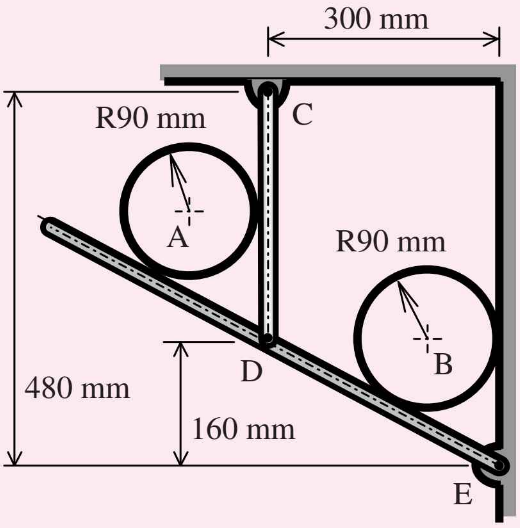
Exercise 12.46

Determine components of the forces acting on member ABC and determine components of the reaction at C and F. All dimensions in mm.



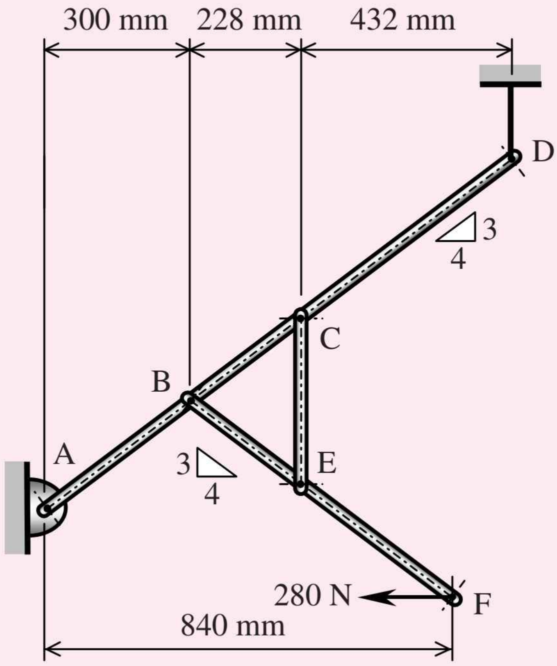
Exercise 12.47

The figure shows two rings, A and B, both with radius 90 mm and mass 4 kg are supported by a frame which consist of members CD and DE. Determine the reactions at C and E. All contacting surfaces are smooth.



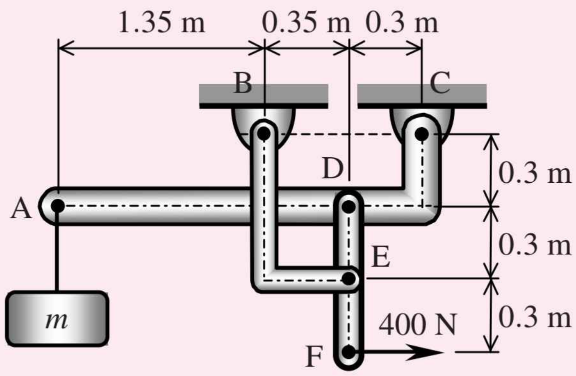
Exercise 12.48

The frame in the figure is loaded with a 280 N force at F, pin jointed at A and supported by a vertical cable at D. Determine all components of forces acting on member ABCD.



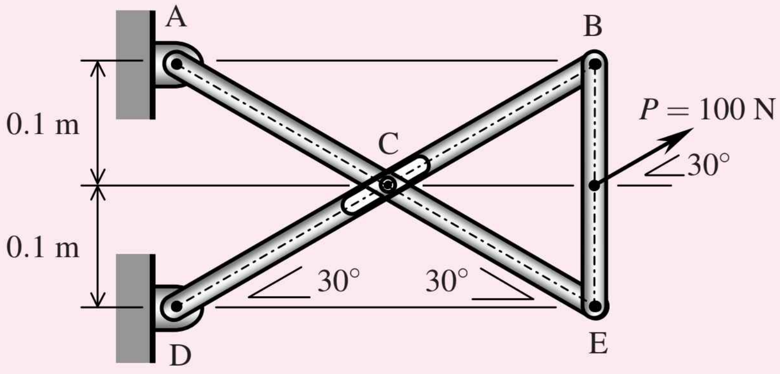
Exercise 12.49

The frame in the figure is used to support the mass m at end A. Determine the mass m (in kg) if a 400 N force is applied at F.



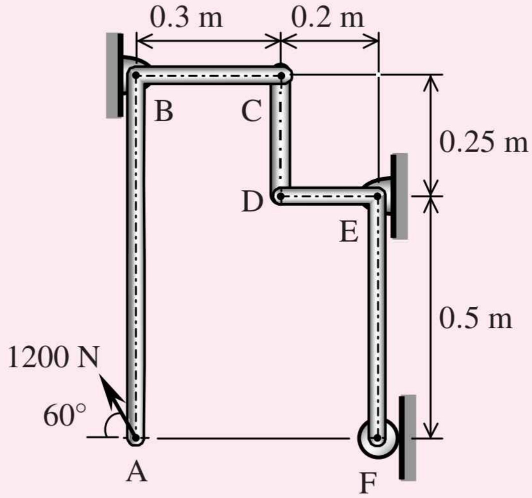
Exercise 12.50

Determine all components of forces acting on member ACE of the frame shown.



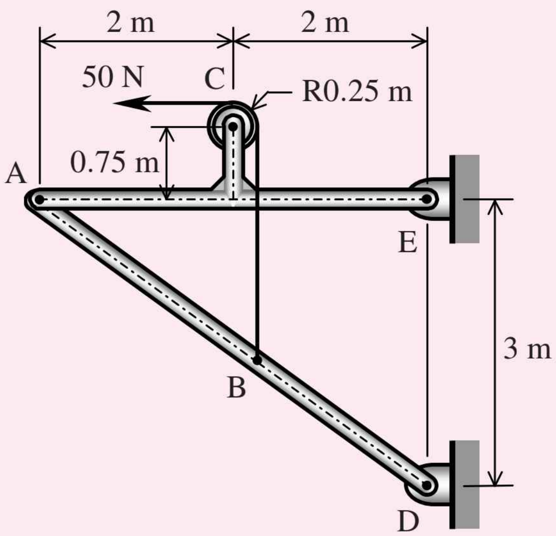
Exercise 12.51

Determine the components of all forces acting on member DEF of the mechanism shown.



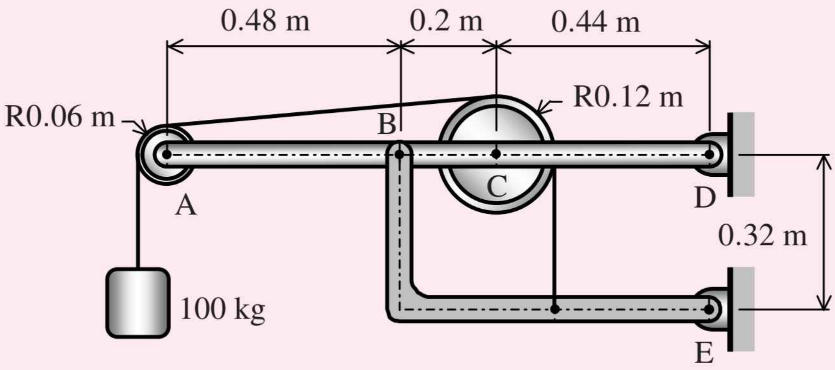
Exercise 12.52

Determine all components of forces acting on member ACE of the frame shown.

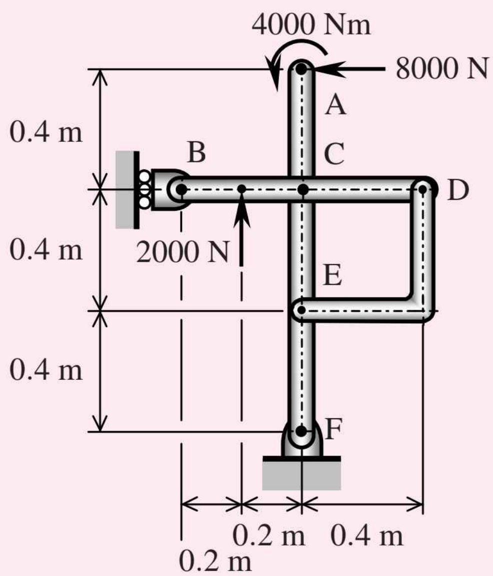


Exercise 12.53

a. Determine all components of forces acting on member ABCD.
b. Determine all components of forces acting on member BE.

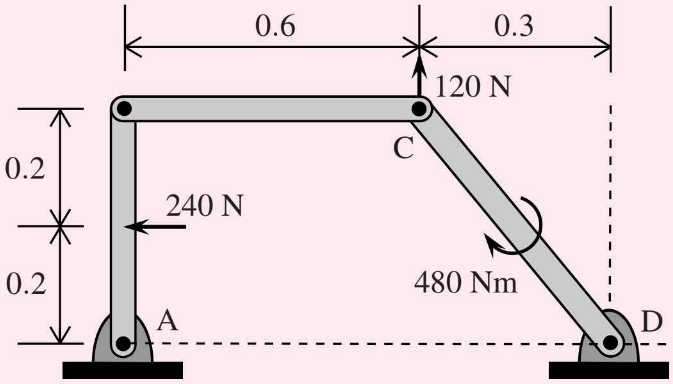


Exercise 12.54
Determine all components of forces acting on member ACEF.



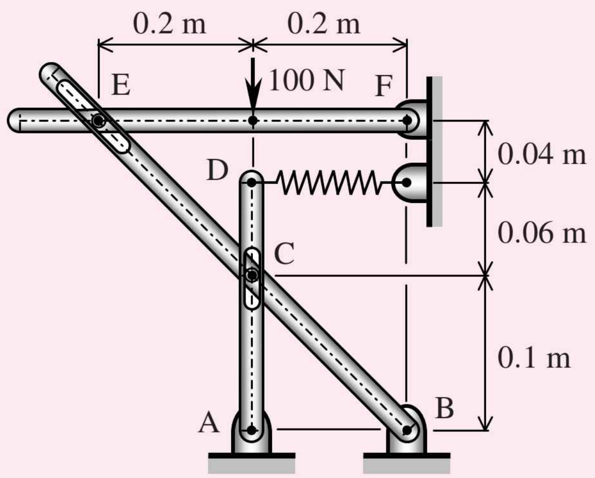
Exercise 12.55

Determine components of the forces acting on member BC and determine components of the reaction at A and D. All dimensions in m.



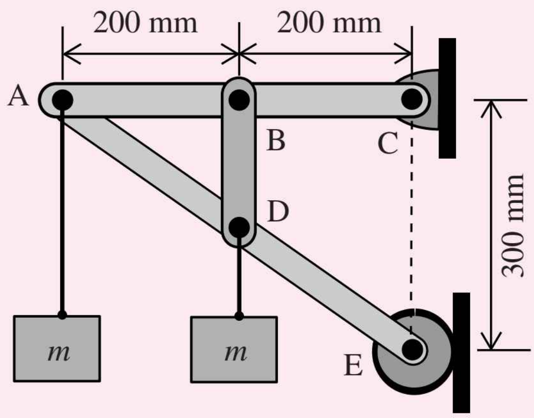
Exercise 12.56

Determine components of the reaction at pins A and B.



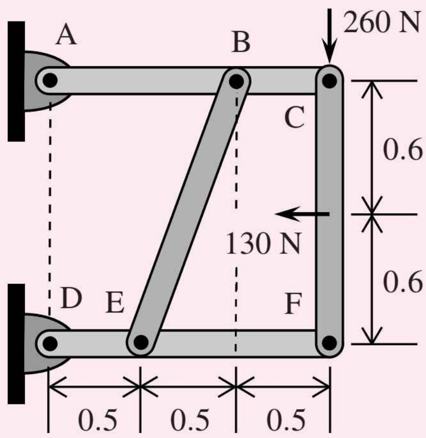
Exercise 12.57

The mass $m = 120$ kg. Determine all components of forces acting on member ABC and ADE.



Exercise 12.58

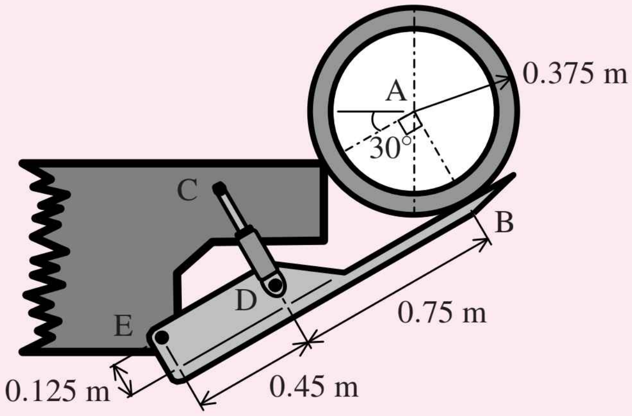
Determine components of the forces acting on member ABC and determine components of the reaction at A and D. All dimensions in m.



12.6 Example questions (machine)

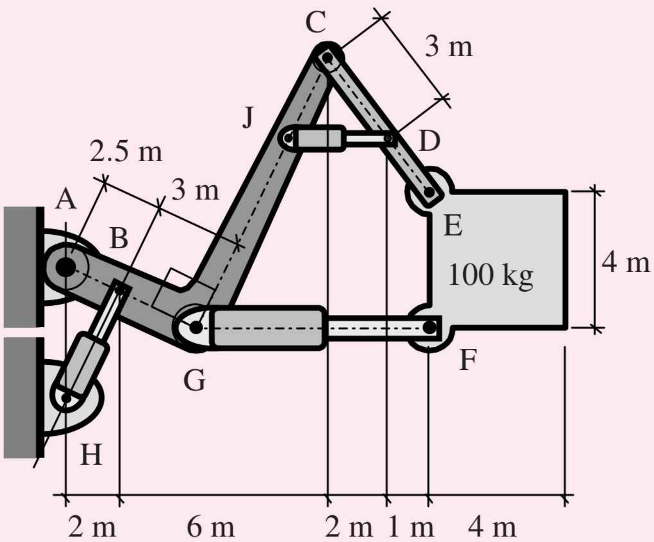
Exercise 12.59

A 1000 kg mass pipe is being lowered from a lorry. Determine the reactions at D and E for the position shown. All contacting surfaces are smooth.



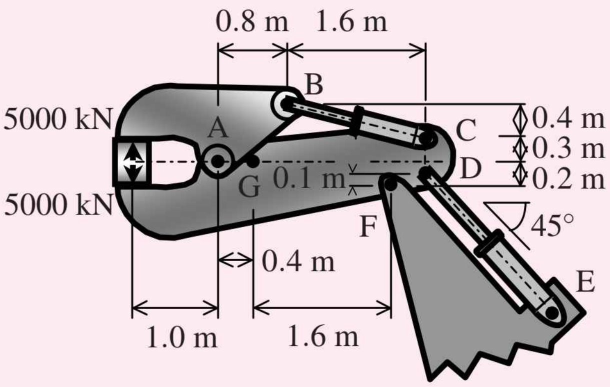
Exercise 12.60

The figure shows a machine used to lift a 100 kg mass. Determine the forces in hydraulic cylinders GF, HB and JD. Determine whether the cylinders are in tension (T) or compression (C).

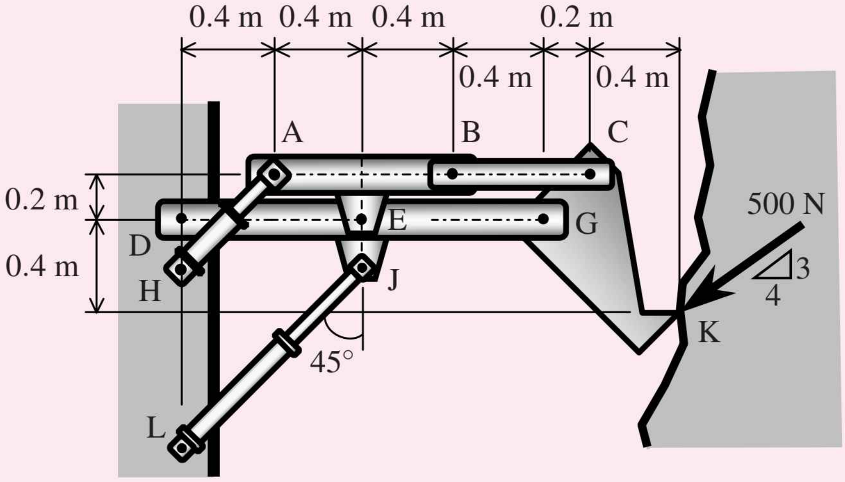


Exercise 12.61

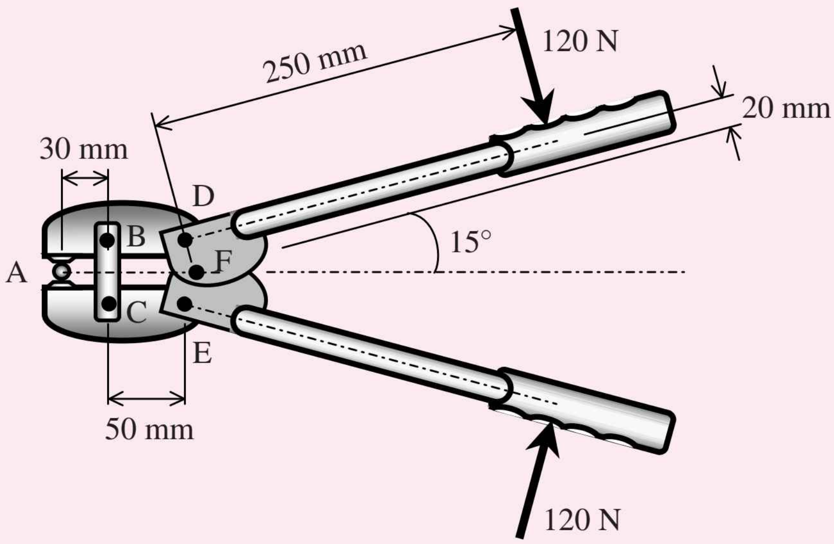
Determine the forces in hydraulic cylinders BC and DE if the work piece exerts a 5000 kN normal force as shown. Weight of the gripping mechanism (excluding member FE and hydraulic cylinder DE) is 500 N and acts at point G.



Exercise 12.62
The figure shows a rock breaking mechanism. Determine all components of forces acting on member ABE if the force exerted by the rock is 5000 N acting at point K. The two hydraulic cylinders are parallel for the position shown. All dimensions are in meters.

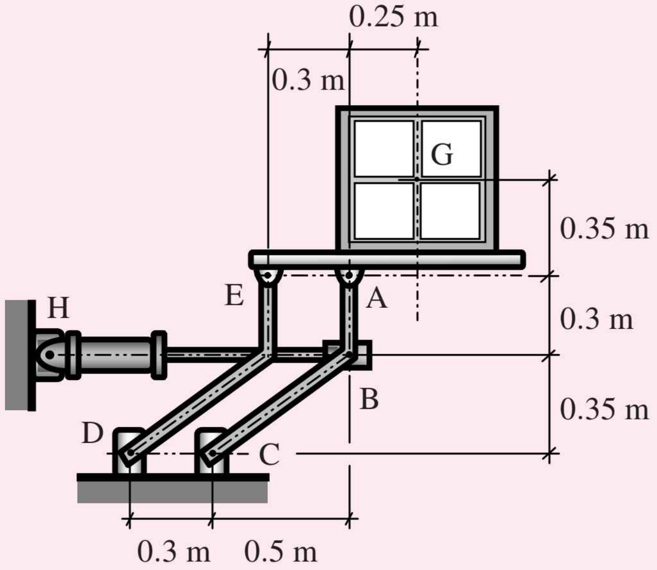


Exercise 12.63
120 N forces are exerted on wire cutter as shown. Determine the forces acting on the wire.



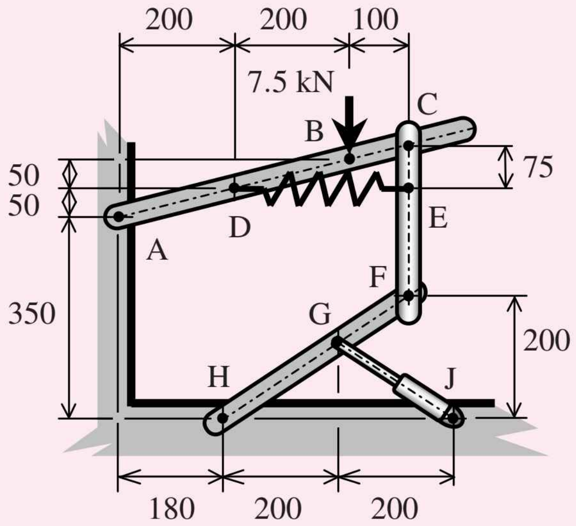
Exercise 12.64

The figure shows a lifting mechanism controlled by the hydraulic cylinder BH. Determine components of the reaction at A and the force in hydraulic cylinder BH to maintain it in the horizontal position. Mass of all other members is negligible compared to the 250 kg mass acting at G. DE is a two-force member.



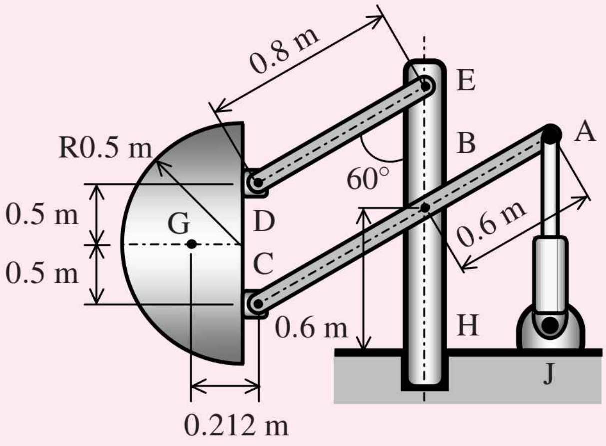
Exercise 12.65

The mechanism shown is used to support the 7.5 kN load. If for the position shown, hydraulic cylinder GJ is neither in tension or compression, determine the force in the spring and state whether it is in tension or compression. Dimensions are in milimeters.



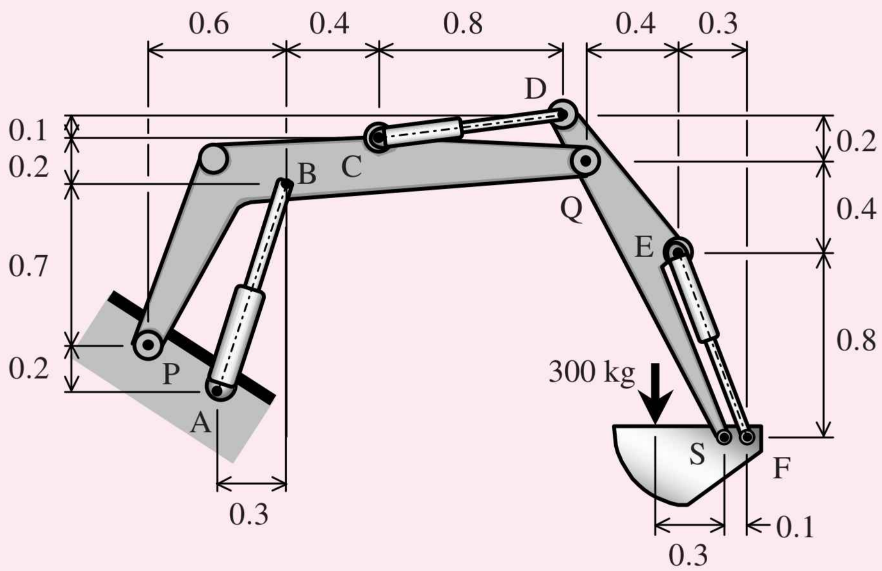
Exercise 12.66

Movement of the 65 kg semicircular plate, with a centre of gravity at G is controlled by the hydraulic cylinder AJ. Members DE and ABC are parallel for the position shown. Determine all components of forces acting on member ABC and the force in hydraulic cylinder AJ. State whether the hydraulic cylinder is in tension or compression.



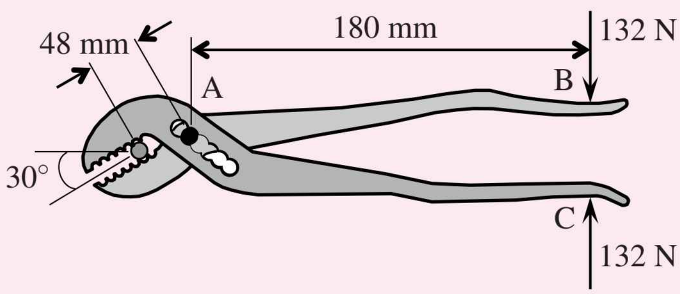
Exercise 12.67

Determine the forces in hydraulic cylinders AB, CD and EF. All dimensions in meters.



Exercise 12.68

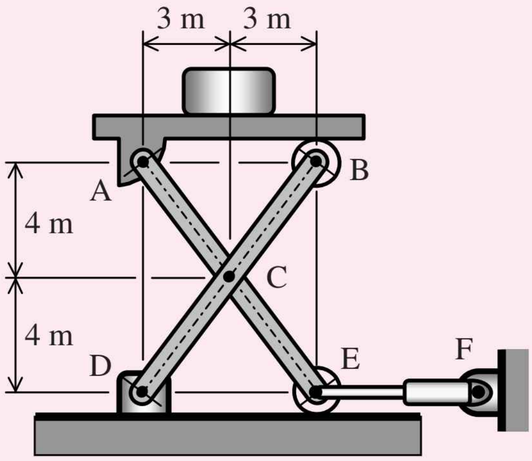
Determine the forces the pliers exert on the smooth bolt.



Exercise 12.69

The mechanism shown is used for to level a load to a desired height. Determine

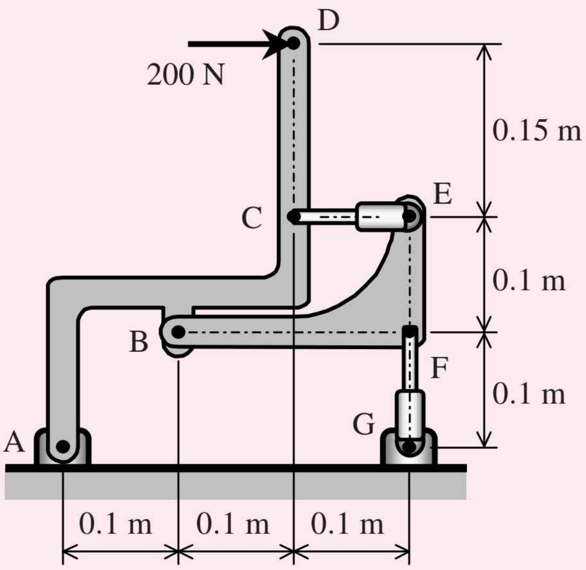
a. the force in hydraulic cylinder EF to level a 3 kN load at the height shown.
b. all components of forces acting on member BCD.



Exercise 12.70

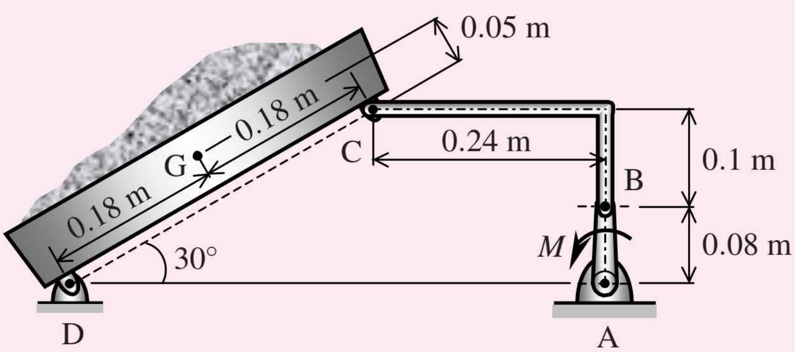
The mechanism shown is used to support the 200 N horizontal force at D by adjusting hydraulic cylinders CE and FG. Determine all components of forces acting on member ABCD for the position shown.

Answer: $A_x = 200$ N(←), $A_y = 233.3$ N(↓), $B_x = 466.7$ N(→), $B_y = 233.3$ N(↑) and $F_{CE} = 466.7$ N(C)

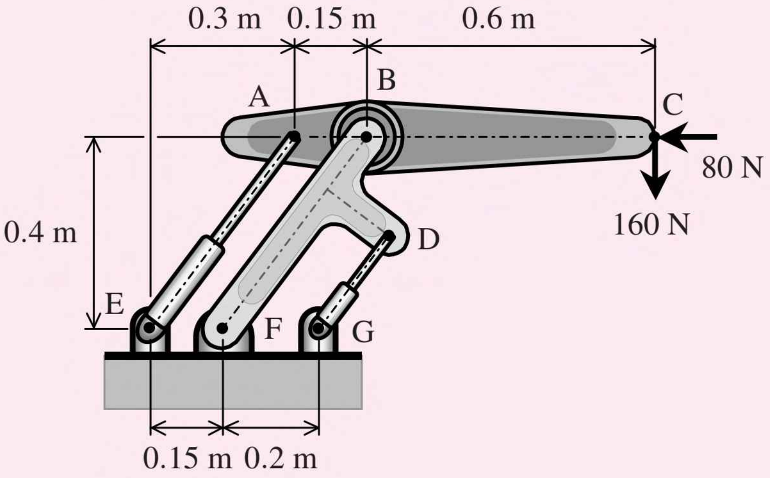


Exercise 12.71

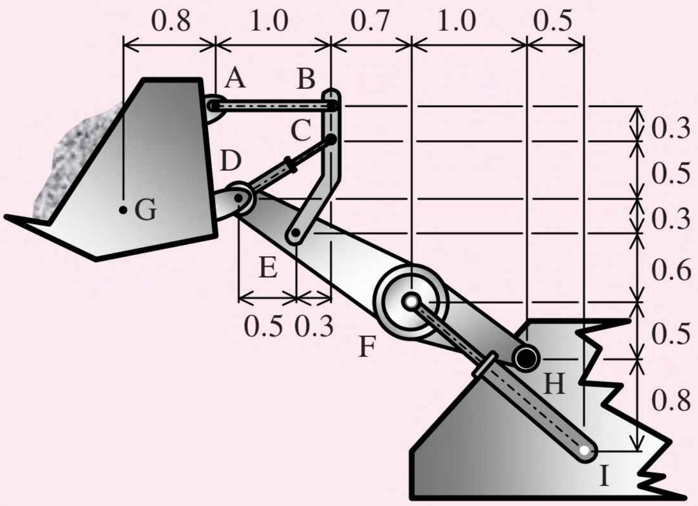
Determine the required moment M to maintain the 1 kN tilting device (acting at G) at the position shown.



Exercise 12.72
Determine the force in hydraulic cylinders AE and DG.

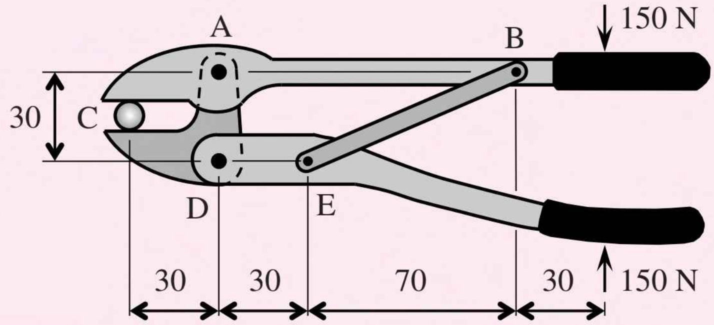


Exercise 12.73
In the particular position shown in the figure, the excavator is supporting a 500 kg load acting at G. Neglecting the weight of other members, determine the force in hydraulic cylinders CD and FI and state whether the cylinders are in tension or compression. All dimensions in meters.



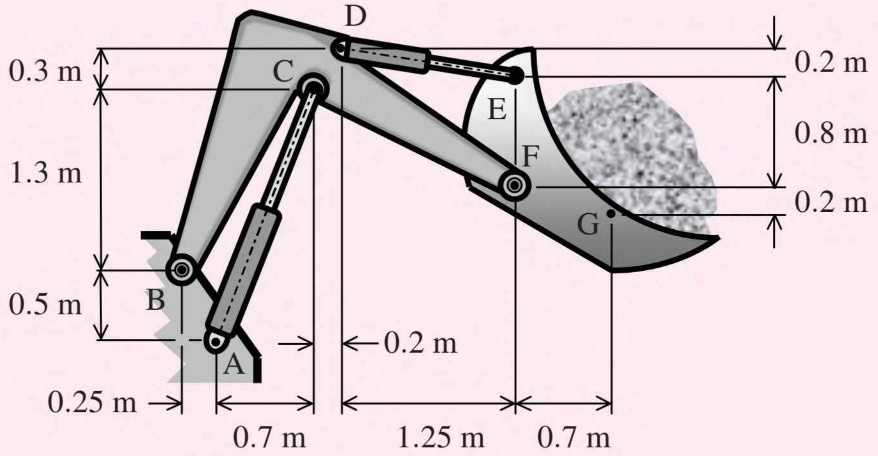
Exercise 12.74

Determine components of forces acting on member AB of the bolt cutters. All dimensions in mm.



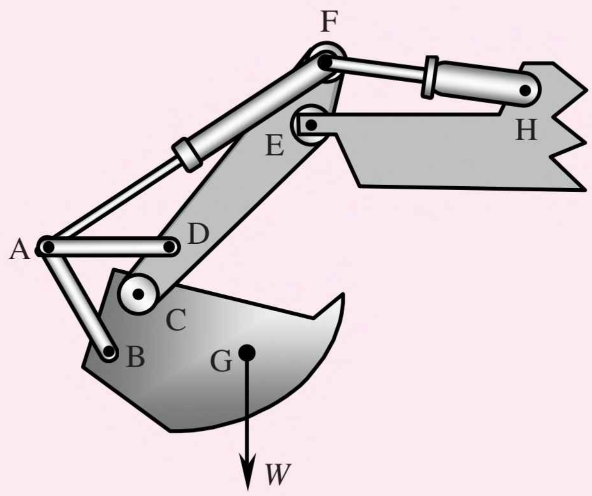
Exercise 12.75

The mechanism shown is used to carry 100 kg of sand acting at G. Determine the forces in hydraulic cylinders AC and DE for the position shown. Determine also components of the reaction at B and F on member BCDF.



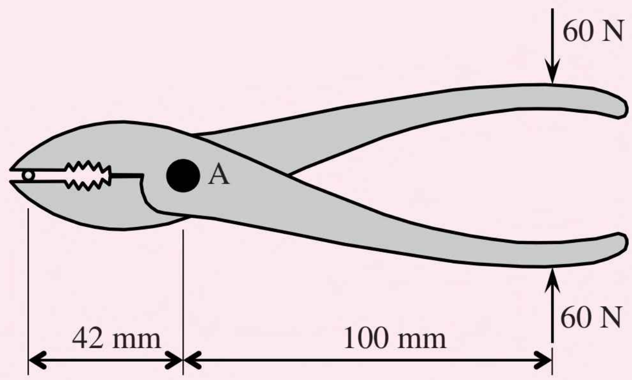
Exercise 12.76

Taking into consideration Newton's Third Law, draw the Free Body Diagrams of ALL members shown in the figure.



Exercise 12.77

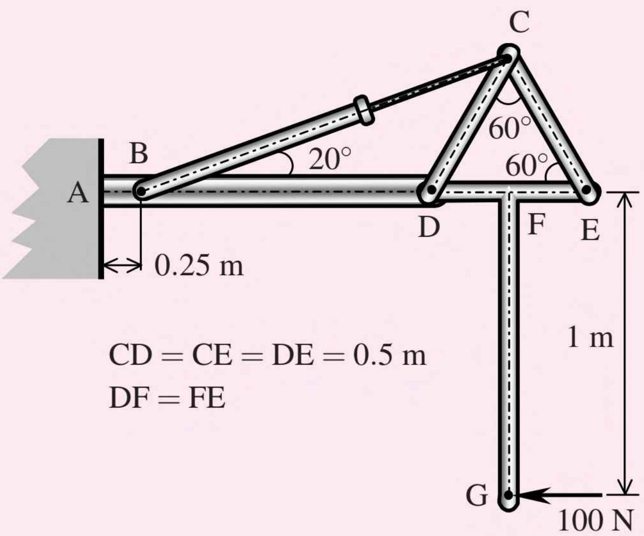
Determine components of the forces the members of the pliers exert on each other at A.



Answer: $A_x = 0$ and $A_y = 202.86$ N(↓)

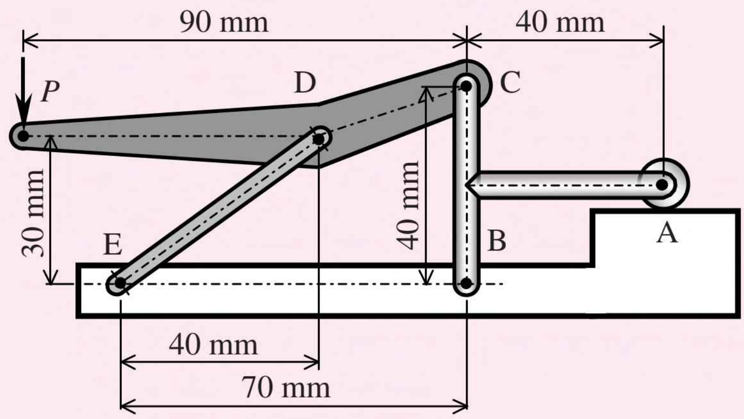
Exercise 12.78

The mechanism shown in the figure is used to support the 100 N load at G. Determine the force acting on the two force member CE and the hydraulic cylinder BC for this instant.



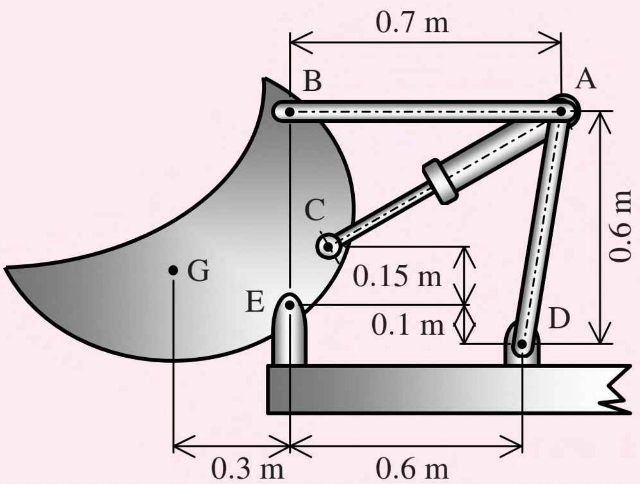
Exercise 12.79

The figure shows a mechanism used to clamp a workpiece at roller A. If the vertical force $P = 100$ N, determine the reaction at roller A for the instant shown.



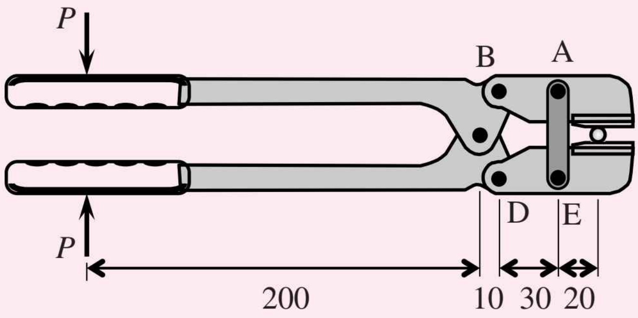
Exercise 12.80

Determine the force in the hydraulic cylinder AC if the mechanism is supporting a load of 10 kN at G.



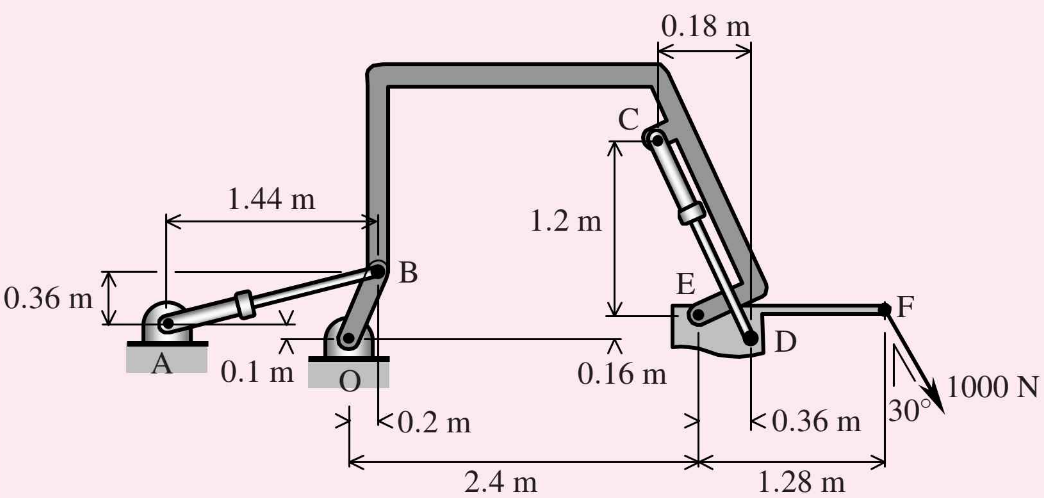
Exercise 12.81

Determine the force exerted on the smooth bolt by the bolt cutters. Force $P = 80$ N. All dimensions in mm.



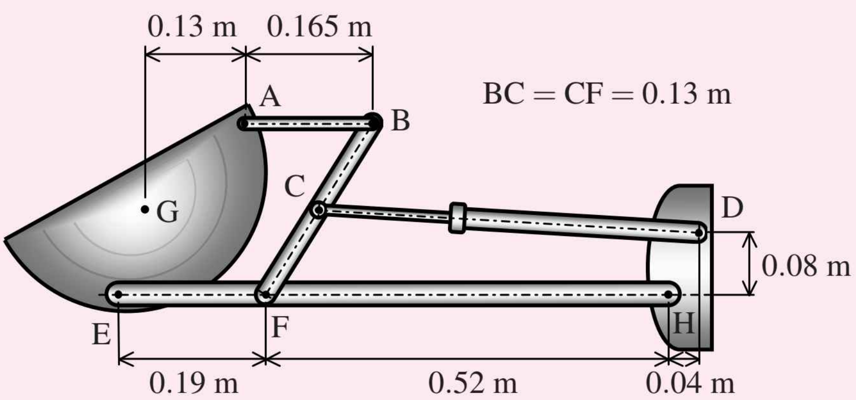
Exercise 12.82

The mechanism shown in the figure is used to raise and rotate the 1000 N load at F. Determine the force required in both hydraulic cylinders to keep the mechanism in the position shown.



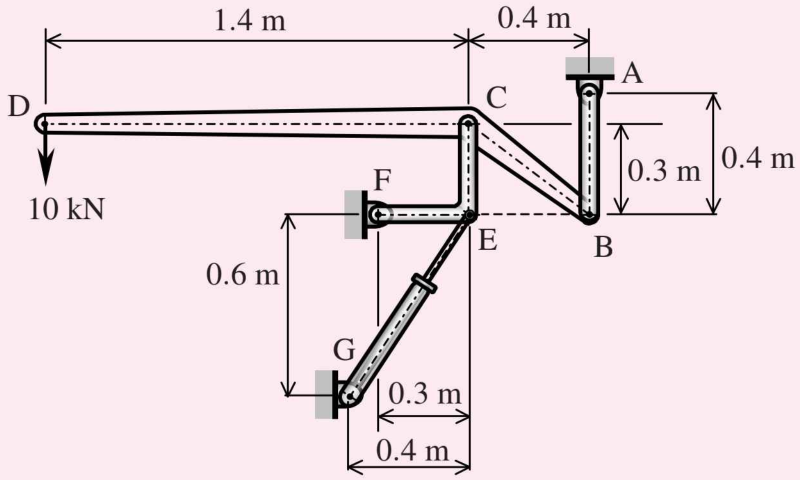
Exercise 12.83

The mechanism shown is supporting a 1 kN load at G. Determine all components of forces acting on member BCF.



Exercise 12.84

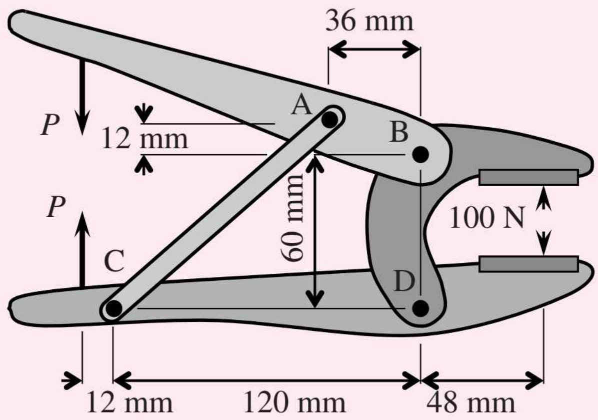
Determine the force acting on the hydraulic cylinder EG if the mechanism is supporting a load of 10 kN at D.



Answer: $F_{EG} = 54$ kN(C)

Exercise 12.85

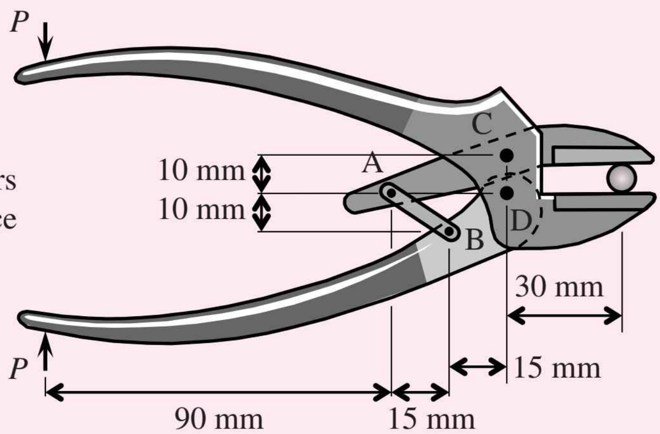
Determine the magnitude of the force P required to produce a 100 N gripping force at the jaws of the vice grip pliers.



Answer: $P = 26$ N

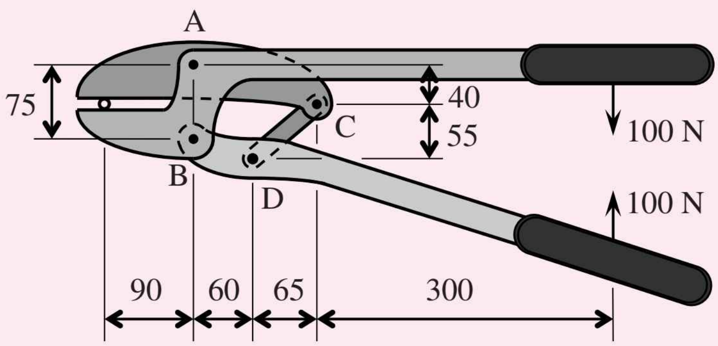
Exercise 12.86

Determine the forces the pliers exert on the smooth bolt. Force $P = 50$ N.



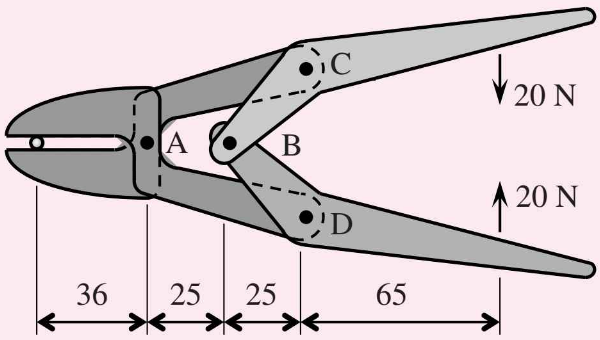
Exercise 12.87

Determine components of the forces acting on member BD of the bolt cutters. All dimensions in mm.



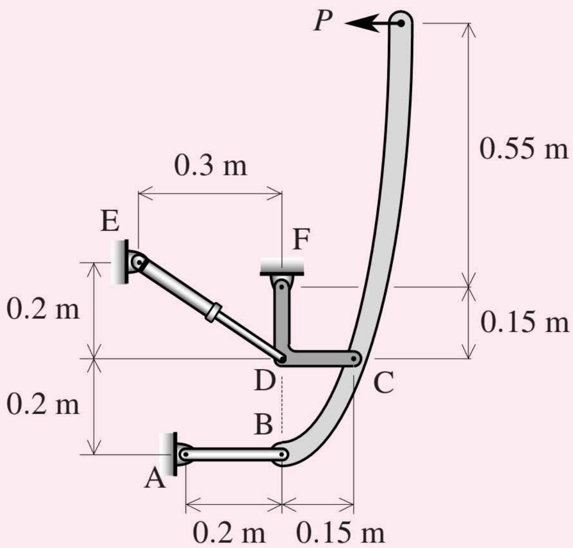
Exercise 12.88

Determine the forces exerted on the twig by the garden shears. All dimensions in mm.



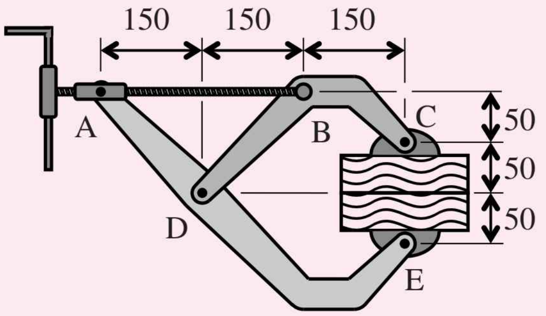
Exercise 12.89

The mechanism shown is used to support the load $P = 8$ kN. Determine component of the reaction at A and F, and the force in hydraulic cylinder DE. State whether the hydraulic cylinder is in tension or compression.



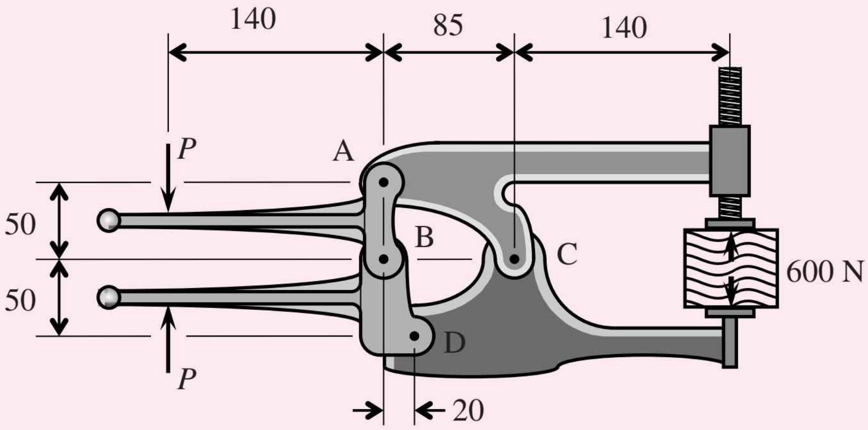
Exercise 12.90

The mechanism is used to clamp two blocks of wood. Determine all components of forces acting on member CBD if the blocks are pressed together with a force of 200 N. All dimensions in mm.



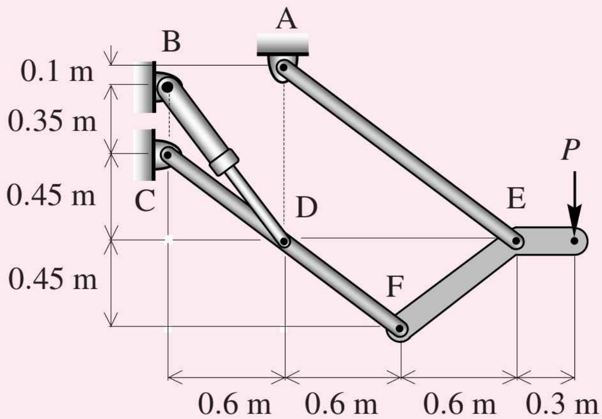
Exercise 12.91

The wooden block exerts a force of 600 N at the clamp. Determine the force P applied at the handle. All dimensions in mm.



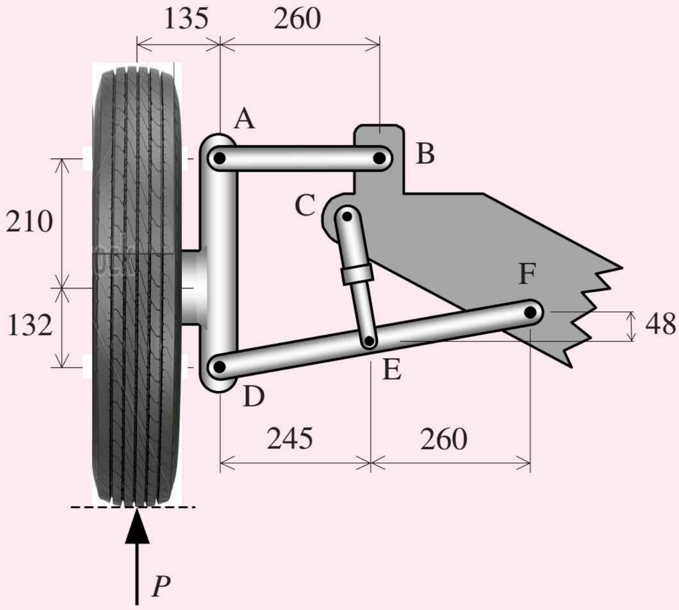
Exercise 12.92

The mechanism shown is used to support the $P = 600$ N load. Determine the reaction at A and the force in hydraulic cylinder BD. State whether the hydraulic cylinder is in tension or compression.



Exercise 12.93

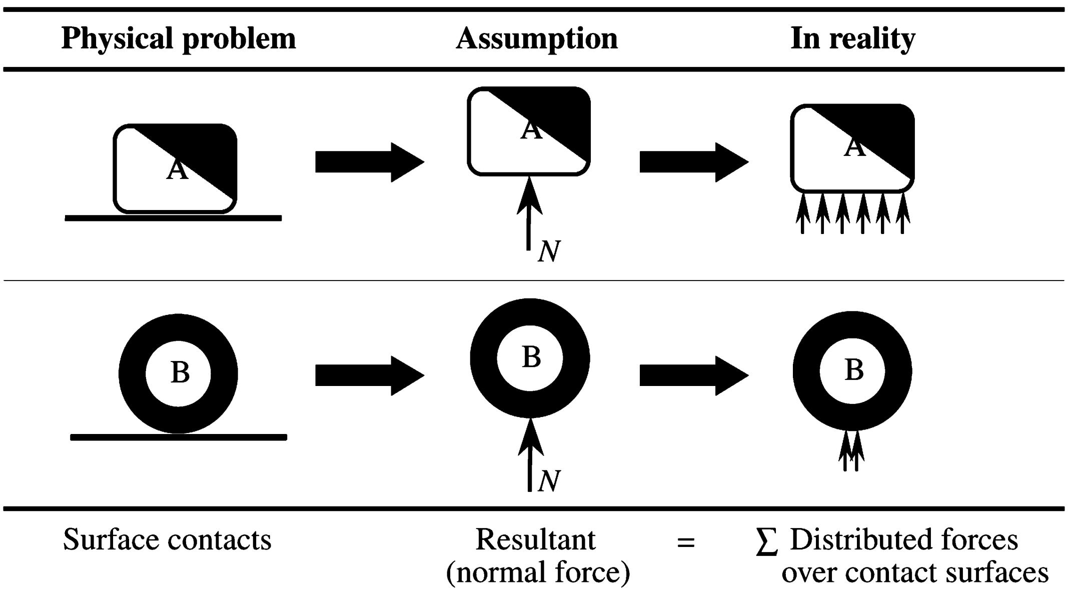
The figure shows the front wheel suspension of an automobile. The pavement reacts a vertical force of $P = 3$ kN on the tyre. Determine the force acting on pin B and components of the force on pin F, and the force on absorber CE. Absorber CE is perpendicular to member DEF. All masses are negligible compared to force P. All dimensions in mm.



13 — Rigid bodies : Distributed forces

13.1 Centroid and centre of gravity

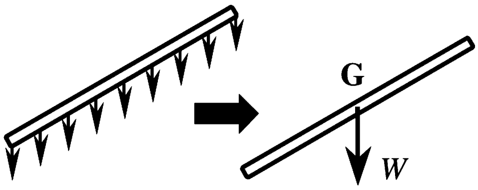
So far we have assumed that the attraction by the earth on a rigid body with a mass m is represented by a single force W (commonly called as weight), applied at the centre of gravity of the body (G). Its magnitude is given by $W = mg$ and is pointing downward (towards the earth). However in reality, this single point force W does not exist. The weight W is actually the resultant of a large number of small distributed forces over the entire body/surface/length. It is easier to illustrate this concept through normal force at contacting surfaces. The normal force is the reaction of W and acts in the opposite direction of of W. Consider contacting surfaces shown below;



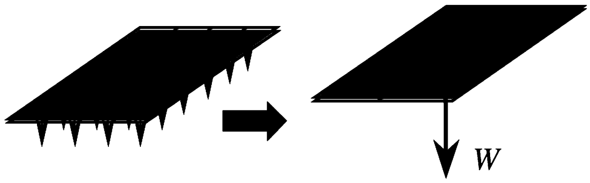
Mass of the body is being distributed depending on its geometry. Hence W due to the distributed mass is coming from one of three categories of distributed forces which are for line, area and volume geometry, respectively as shown below;

Idealization for three categories of distributed forces due to distributed mass

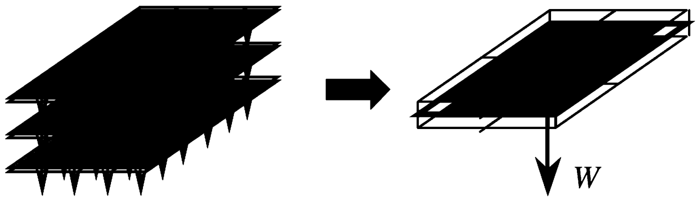
1. Line distribution



2. Area distribution



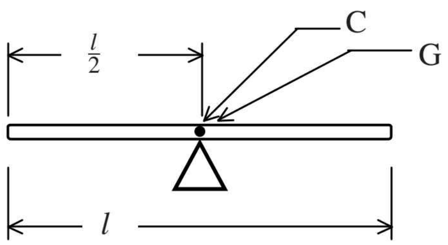
3. Volume distribution



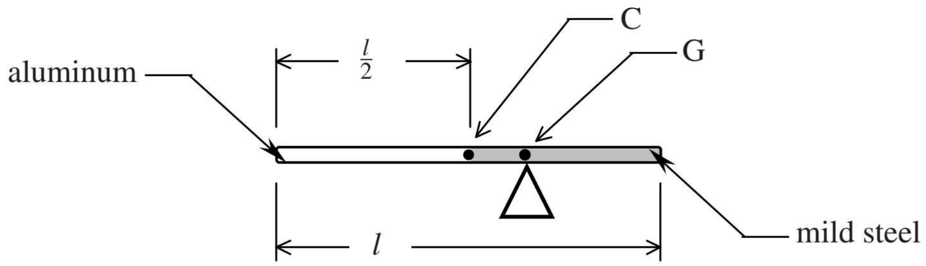
Weight W is acting at the centre of gravity (G) of a body. The concept of centroid (C) and centre of gravity (G) can be better understood by the illustration of two case studies as follows. In simple words, C is the center of shape (refer to area) while G is the center of mass (refer to volume). However, for a homogeneous thin plate for example, the location of C and G is the same.

■ **Example 13.1** **Case study 1**

A homogeneous rod of length l is made of aluminium. The centroid (C) which is the centre of the shape is located at the mid point ($=\frac{l}{2}$). The centre of gravity (G) is also located at the mid point.

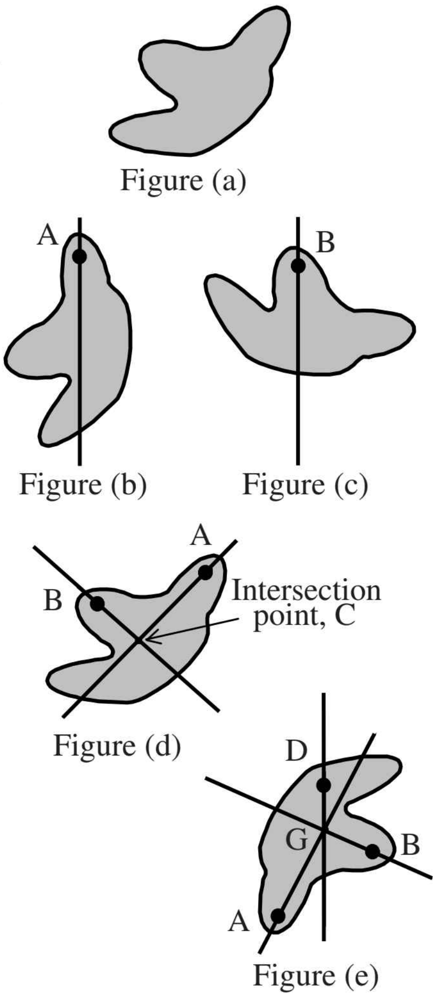


A second homogeneous rod of length l is made of two parts of equal length, half aluminium and half mild steel. The centroid (C) is still located at the mid point ($=\frac{l}{2}$). However, the centre of gravity (G) is no longer located at the mid point.



■ **Example 13.2** **Case study 2**

This is an example for two dimensional analysis. Consider a homogeneous plate with uniform thickness having an arbitrary shape as shown in Figure (a). How to determine the centre of gravity (G) of the plate? Since the plate is homogeneous and of uniform thickness, both the centroid (C) and the centre of gravity (G) is located at the same point.



<u>Procedure to locate points C and G</u>

1. Mark a point on the plate, i.e point A
2. Hinge the plate about point A (Figure (b))
3. Construct a vertical line passing point A
4. Do steps (1)-(3) as above for point B
5. Intersection points of the two lines is the centroid (C) as shown in Figure (d).

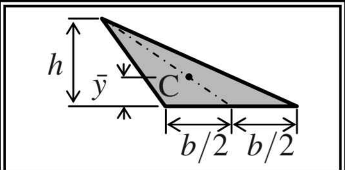
If another point is chosen (i.e. point D), the vertical line constructed will pass through the centroid C (which is also the point G), as shown in Figure (e). The results achieved enhance the concept that;

Moment of area on the left side of the vertical line	=	Moment of area on the right side of the vertical line

13.2 Centroid for common shape areas and lines

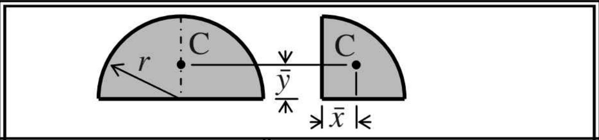
The scope of this subject is more inclined towards the ability to solve application rather than derivation problems. In view of this, a collection of centroids of common shapes of areas and lines is presented to assist solutions. The centroid for typical shapes are shown in Table 13.1. The 2D shapes represent homogeneous plates with constant thickness. As the point for the centroid coincide with the center of gravity for 2D shapes, these terms will be used interchangeably.

Triangular area



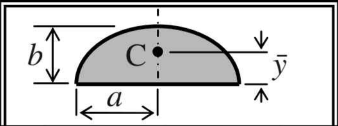
$A = \frac{bh}{2}$ $\bar{y} = \frac{h}{3}$

Quarter and semi-circular area



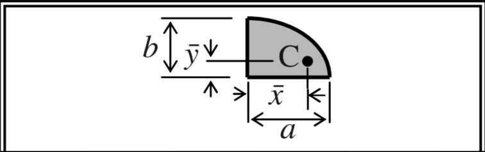
$A = \frac{\pi r^2}{2}$ $\bar{y} = \frac{4r}{3\pi}$ | $A = \frac{\pi r^2}{4}$ $\bar{x} = \bar{y} = \frac{4r}{3\pi}$

Semi-eliptical area



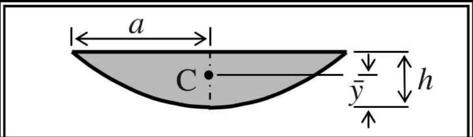
$A = \frac{\pi ab}{2}$ $\bar{y} = \frac{4b}{3\pi}$

Quarter eliptical area



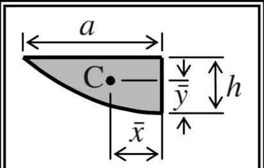
$A = \frac{\pi ab}{4}$ $\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$

Parabolic area



$A = \frac{2ah}{3}$ $\bar{y} = \frac{3h}{5}$

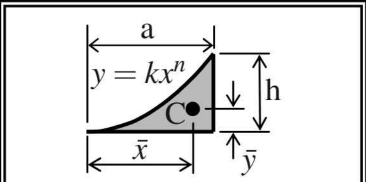
Semi-parabolic area



$A = \frac{4ah}{3}$

$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3h}{5}$

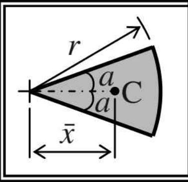
General spandrel



$A = \frac{ah}{n+1}$

$\bar{x} = (\frac{n+1}{n+2})a$ $\bar{y} = (\frac{n+1}{4n+2})h$

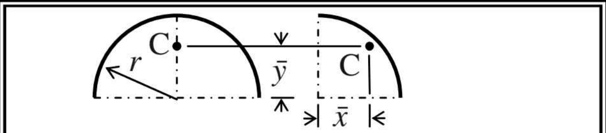
Circular sector



$A = ar^2$

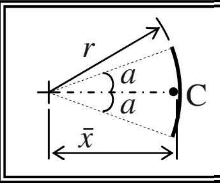
$\bar{x} = \frac{2r\sin a}{3a}$

Quarter and semi-circular arc



$L = \pi r$ $\bar{y} = \frac{2r}{\pi}$ | $L = \frac{\pi r}{2}$ $\bar{x} = \bar{y} = \frac{2r}{\pi}$

Arc of circle



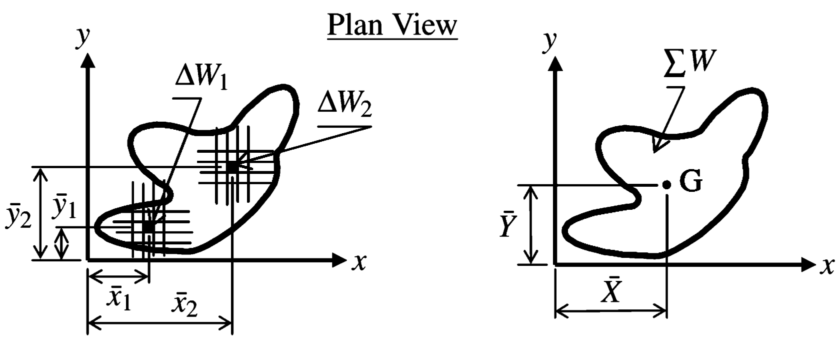
$L = 2ar$

$\bar{x} = \frac{r\sin a}{a}$

Table 13.1: Centroid for common shape areas and lines.

13.2.1 Centroid/center of gravity for common and composite shape areas

Consider a homogeneous plate with uniform thickness on x-y plane. Since the plate has homogeneous thickness, only x and y axes need to be considered. Furthermore, the centroid and center of gravity of this plate are located on the same point.



For determining the centre of gravity of a homogeneous plate with uniform thickness about x-axis, moment is taken along y-axis as;

$$\begin{aligned} M_y &= \text{moment of distributed forces} = \text{moment of the resultant} \\ &= \bar{x}_1 \Delta W_1 + \bar{x}_2 \Delta W_2 + \bar{x}_3 \Delta W_3 + \ldots = \bar{X} \sum \Delta W \\ &= \sum \bar{x} \Delta W = \bar{X} \sum \Delta W \end{aligned}$$

Therefore the coordinate of point G about the x-axis is

$$\bar{X} = \frac{\sum \bar{x} \Delta W}{\sum \Delta W}$$

and by using the same procedure, the coordinate of point G about the y-axis is given as

$$\bar{Y} = \frac{\sum \bar{y} \Delta W}{\sum \Delta W}$$

where

$\sum W$	=	overall weight
ΔW	=	weight of element
$\bar{x}$	=	c.g. of element (x-axis)
$\bar{y}$	=	c.g. of element (y-axis)
$\bar{X}$	=	c.g. of plate (x-axis)
$\bar{Y}$	=	c.g. of plate (y-axis)

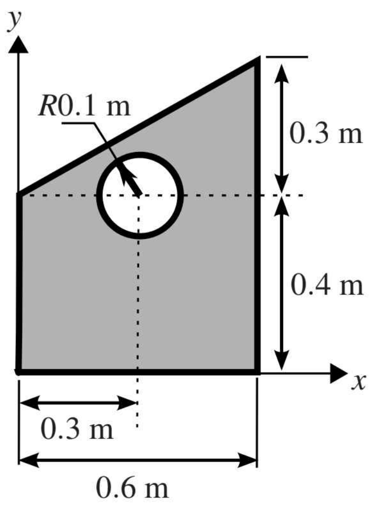
Knowing that $W = \rho g t A$, therefore $\Delta W = \rho g t \Delta A$ with ρ is density of the plate, $g = 9.81\ \mathrm{m/s^2}$, t is the plate thickness, A is the plate area and ΔA is the element area.

For a homogenous plate, and constant ρ, g and t, the equation can be simplified to obtain the coordinate of the center of gravity (G) about the x-axis and y-axis as

$$\bar{X} = \frac{\sum \bar{x} \Delta A}{\sum \Delta A} \quad ; \qquad \bar{Y} = \frac{\sum \bar{y} \Delta A}{\sum \Delta A} \tag{13.1}$$

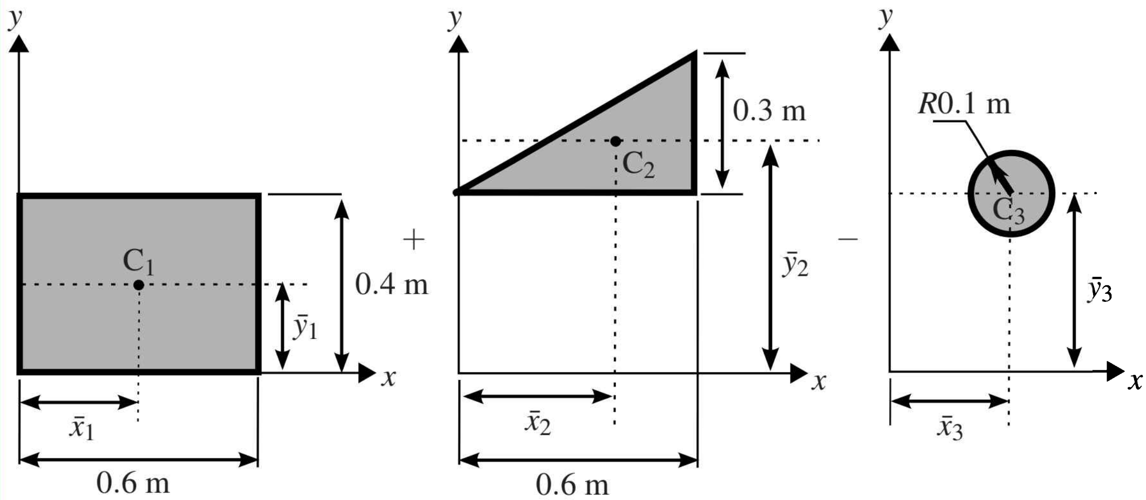
■ Example 13.3

Determine the location of centroid of the composite body shown in the diagram



Solution:

Step 1	Determine the axes.
Step 2	Segregation into simple shapes. Ensure that combination of segregated parts will produce the original body. Areas of holes must be subtracted.
Step 3	Calculate the area (A) of each component.
Step 4	Determine the $(\bar{x}, \bar{y})$ coordinates of each component of simple shapes.
Step 5	Construct a table.
Step 6	Solve the coordinate of G for the composite body using $\bar{X} = \frac{\sum \bar{x}\Delta A}{\sum \Delta A}$ and $\bar{Y} = \frac{\sum \bar{y}\Delta A}{\sum \Delta A}$.



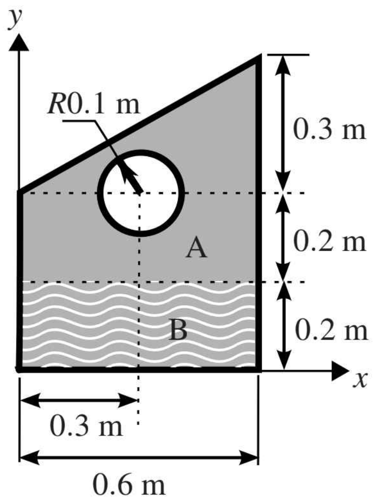
Part	$\Delta A(\mathrm{m}^2)$	$\bar{x}(\mathrm{m})$	$\bar{y}(\mathrm{m})$	$\bar{x}\Delta A(\mathrm{m}^3)$	$\bar{y}\Delta A(\mathrm{m}^3)$
1	$(0.4)(0.6)$	0.3	0.2	0.072	0.048
2	$\frac{1}{2}(0.6)(0.3)$	$\frac{2}{3}(0.6)$	$0.4 + \frac{1}{3}(0.3)$	0.036	0.045
3	$-\pi(0.1)^2$	0.3	0.4	−0.00942	−0.01256
	$\sum \Delta A = 0.2986$			$\sum \bar{x}\Delta A = 0.09838$	$\sum \bar{y}\Delta A = 0.08044$

$$\therefore \bar{X} = \frac{\sum \bar{x}\Delta W}{\sum \Delta W} = \frac{\sum \bar{x}\Delta A}{\sum \Delta A} = \frac{0.09838}{0.2986} = 0.329 \text{ m}$$

$$\therefore \bar{Y} = \frac{\sum \bar{y}\Delta W}{\sum \Delta W} = \frac{\sum \bar{y}\Delta A}{\sum \Delta A} = \frac{0.08044}{0.2986} = 0.269 \text{ m}$$

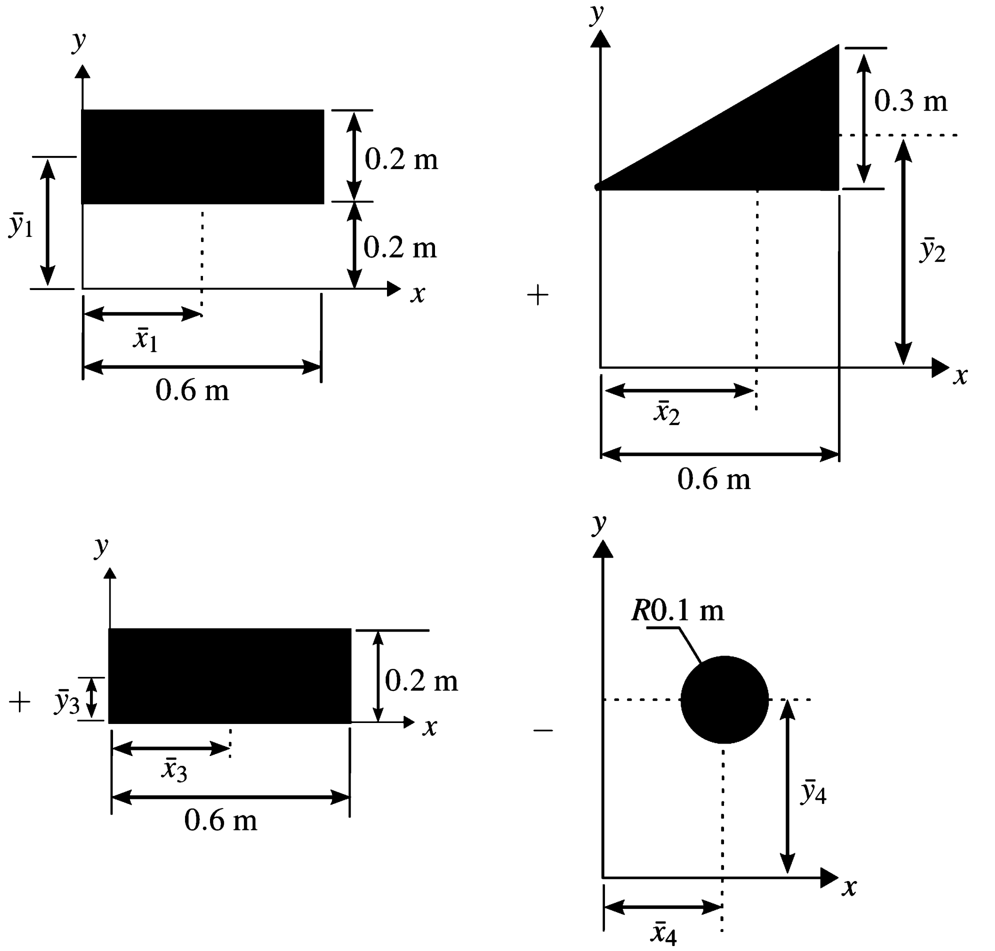
■ Example 13.4

Determine the location of the centre of gravity of the composite body shown given that the density of material A is twice that of material B.



Solution:

Step 1	Determine the axes.
Step 2	Segregation into simple shapes. Ensure that combination of segregated parts will produce the original body. Areas of holes must be subtracted.
Step 3	Calculate the weight (W) of each component.
Step 4	Determine the $(\bar{x}, \bar{y})$ coordinates of each component of simple shapes.
Step 5	Construct a table.
Step 6	Solve the coordinate of G for the composite body using $\bar{X} = \frac{\sum \bar{x}\Delta W}{\sum \Delta W}$ and $\bar{Y} = \frac{\sum \bar{y}\Delta W}{\sum \Delta W}$.



From previously, $W = \rho g t A$ and $\Delta W = \rho g t \Delta A$. For a constant g and t, analysis of (ρ and A) and (ρ and ΔA) are done in place of W and ΔW. Given that $\rho_A = 2\rho_B$, therefore;

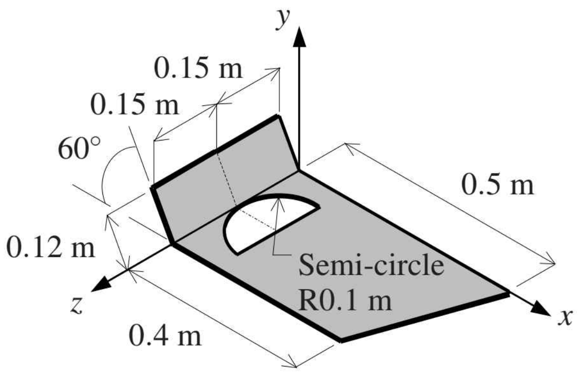
Part	$\rho\Delta A$	$\bar{x}$(m)	$\bar{y}$(m)	$\bar{x}\rho\Delta A$	$\bar{y}\rho\Delta A$
1	$(0.2)(0.6)\rho_A$	0.3	0.3	$0.036\rho_A$	$0.036\rho_A$
2	$\frac{1}{2}(0.6)(0.3)\rho_A$	$\frac{2}{3}(0.6)$	$0.4 + \frac{1}{3}(0.3)$	$0.036\rho_A$	$0.045\rho_A$
3	$(0.2)(0.6)\rho_B = 0.06\rho_A$	0.3	0.1	$0.018\rho_A$	$0.006\rho_A$
4	$-\pi(0.1)^2\rho_A$	0.3	0.4	$-0.00942\rho_A$	$-0.01256\rho_A$
	$\sum\rho\Delta A = 0.2386\rho_A$			$\sum\bar{x}\rho\Delta A = 0.0806\rho_A$	$\sum\bar{y}\rho\Delta A = 0.0857\rho_A$

$$\therefore \bar{X} = \frac{\sum\bar{x}\Delta W}{\sum\Delta W} = \frac{\sum\bar{x}\rho\Delta A}{\sum\rho\Delta A} = \frac{0.0806\rho_A}{0.2386\rho_A} = 0.338 \text{ m}$$

$$\therefore \bar{Y} = \frac{\sum\bar{y}\Delta W}{\sum\Delta W} = \frac{\sum\bar{y}\rho\Delta A}{\sum\rho\Delta A} = \frac{0.0857\rho_A}{0.2386\rho_A} = 0.359 \text{ m}$$

■ Example 13.5

Determine the centroid of the composite sheet shown.



Solution:

Part	ΔA	$\bar{x}$(m)	$\bar{y}$(m)	$\bar{z}$(m)	$\bar{x}\Delta A$	$\bar{y}\Delta A$	$\bar{z}\Delta A$
1	(0.4)(0.3)	0.2	0	0.15	0.024	0	0.018
2	$\frac{1}{2}(0.3)(0.1)$	$0.4 + \frac{1}{3}(0.1)$	0	$\frac{1}{3}(0.3)$	0.0065	0	0.0015
3	(0.12)(0.3)	$-0.06\sin 30°$	$0.06\cos 30°$	0.15	0.00108	1.87E-3	0.0054
4	$-\frac{\pi}{2}(0.1)^2$	$0.1 - \frac{4(0.1)}{3\pi}$	0	0.15	−9.04E-4	0	−2.36E-3
	$\sum\Delta A = 0.1553$				$\sum\bar{x}\Delta A = 0.0307$	$\sum\bar{y}\Delta A$ 0.0019	$\sum\bar{z}\Delta A$ 0.0716

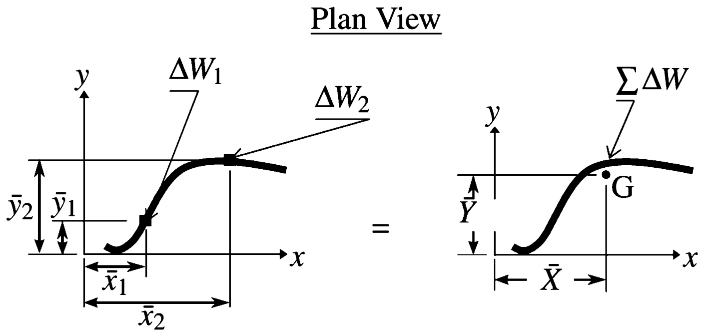
$$\therefore \bar{X} = \frac{\sum \bar{x}\Delta A}{\sum \Delta A} = \frac{0.0307}{0.1553} = 0.1977 \text{ m}$$

$$\therefore \bar{Y} = \frac{\sum \bar{y}\Delta A}{\sum \Delta A} = \frac{0.0019}{0.1553} = 0.012 \text{ m}$$

$$\therefore \bar{Z} = \frac{\sum \bar{z}\Delta A}{\sum \Delta A} = \frac{0.0716}{0.1553} = 0.461 \text{ m}$$

13.2.2 Centroid/center of gravity for common and composite lines

The centroid and centre of gravity of a homogeneous line with uniform cross-sectional area is determined as follows;



Taking moment along x-axis gives

$$M_y = \text{moment of distributed force} = \text{moment of the resultant} \tag{13.2}$$

$$= \bar{x}_1\Delta W_1 + \bar{x}_2\Delta W_2 + \bar{x}_3\Delta W_3 + \ldots = \bar{X}\sum \Delta W \tag{13.3}$$

hence

$$\sum \bar{x}\Delta W = \bar{X}\sum \Delta W$$

giving the center of gravity (G) about the x-axis as;

$$\therefore \bar{X} = \frac{\sum \bar{x}\Delta W}{\sum \Delta W}$$

and by using the same procedure, the center of gravity (G) about the y-axis is;

$$\therefore \bar{Y} = \frac{\sum \bar{y}\Delta W}{\sum \Delta W}$$

where

$\sum W$	=	overall weight
ΔW	=	weight of element
$\bar{x}$	=	c.g. of element (x-axis)
$\bar{y}$	=	c.g. of element (y-axis)
$\bar{X}$	=	c.g. of plate (x-axis)
$\bar{Y}$	=	c.g. of plate (y-axis)

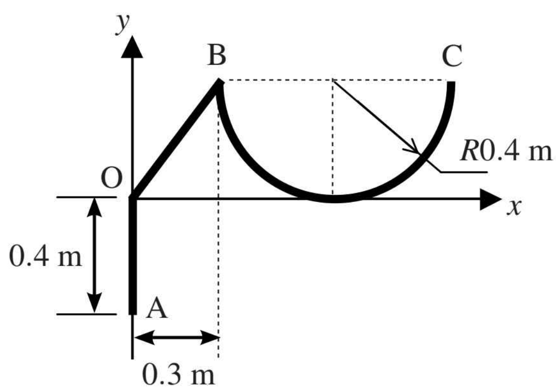
Knowing that $W = \rho g a L$, therefore $\Delta W = \rho g a \Delta L$ with ρ is density of the line, $g = 9.81\ \text{m}/\text{s}^2$, a is the cross-sectional area, L is the line length and ΔL is the element length.

For a homogenous line, and constant ρ, g and a, the equation can be simplified to obtain the coordinate of the center of gravity (G) about the x-axis and y-axis as

$$\bar{X} = \frac{\sum \bar{x}\Delta L}{\sum \Delta L} \quad ; \qquad \bar{Y} = \frac{\sum \bar{y}\Delta L}{\sum \Delta L} \tag{13.4}$$

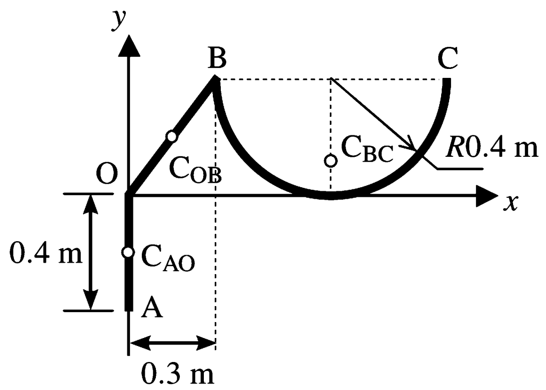
■ Example 13.6

Determine the location of centroid of the composite line shown in the diagram.



Solution:

Step 1	Determine the axes.
Step 2	Segregation into common line segments.
Step 3	Calculate the length (ΔL) of each segments.
Step 4	Determine the $(\bar{x}, \bar{y})$ coordinates of each segment.
Step 5	Construct a table.
Step 6	Solve the coordinate of G for the composite body using $\bar{X} = \frac{\sum \bar{x}\Delta L}{\sum \Delta L}$ and $\bar{Y} = \frac{\sum \bar{y}\Delta L}{\sum \Delta L}$.



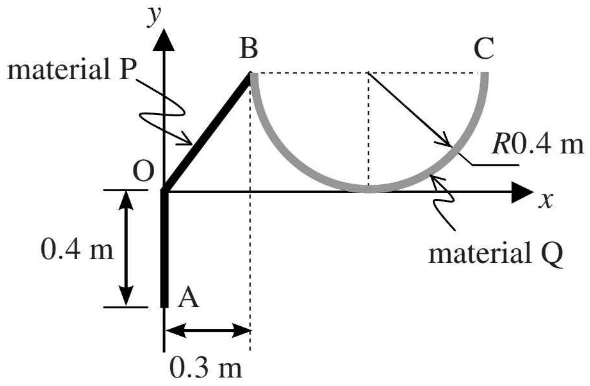
Part	ΔL(m)	$\bar{x}$(m)	$\bar{y}$(m)	$\bar{x}\Delta L$(m^2)	$\bar{y}\Delta L$(m^2)
AO	(0.4)	0	-0.2	0	-0.08
OB	$\sqrt{(0.4)^2+(0.3)^2} = 0.5$	0.15	0.2	0.075	0.1
BC	$\pi(0.4) = 1.257$	$0.3+0.4 = 0.7$	$0.4 - 2(0.4)/\pi = 0.1453$	0.88	0.1826
	$\sum \Delta L = 2.157$			$\sum \bar{x}\Delta L = 0.955$	$\sum \bar{y}\Delta L = 0.2026$

$$\therefore \bar{X} = \frac{\sum \bar{x}\Delta W}{\sum \Delta W} = \frac{\sum \bar{x}\Delta L}{\sum \Delta L} = \frac{0.955}{2.157} = 0.4427 \text{ m}$$

$$\therefore \bar{Y} = \frac{\sum \bar{y}\Delta W}{\sum \Delta W} = \frac{\sum \bar{y}\Delta L}{\sum \Delta L} = \frac{0.2026}{2.157} = 0.0939 \text{ m}$$

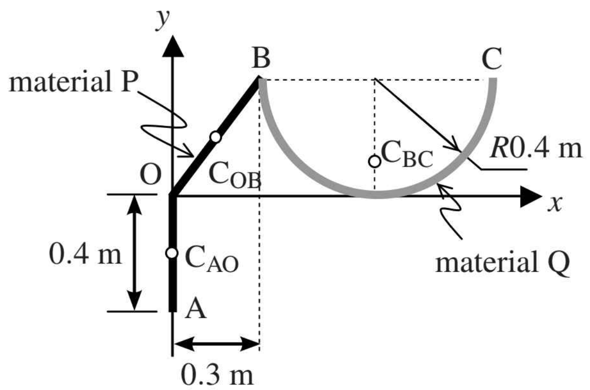
■ Example 13.7

Determine the location of the centre of gravity of the composite line shown, given that the density of materials P and Q is govern by the ratio 4:1.



Solution:

Step 1	Determine the axes.
Step 2	Segregation into common line segments.
Step 3	Calculate the weight (ΔW) of each segments.
Step 4	Determine the $(\bar{x}, \bar{y})$ coordinates of each segment.
Step 5	Construct a table.
Step 6	Solve the coordinate of G for the composite body using $\bar{X} = \frac{\sum \bar{x}\Delta W}{\sum \Delta W}$ and $\bar{Y} = \frac{\sum \bar{y}\Delta W}{\sum \Delta W}$.



From previously, $W = \rho g a L$ and $\Delta W = \rho g a \Delta L$. For a constant g and a, analysis of (ρ and L) and (ρ and ΔL) are done in place of W and ΔW. Given that $\rho_P : \rho_Q = 4:1$, therefore $\rho_P = 4\rho_Q$.

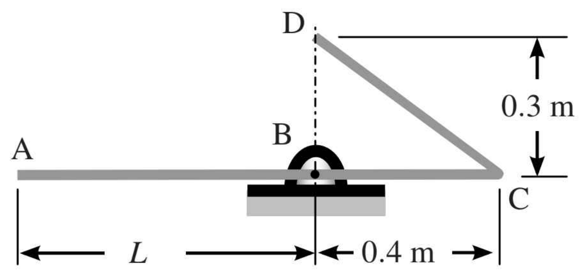
Part	$\rho\Delta L$	$\bar{x}$(m)	$\bar{y}$(m)	$\bar{x}\rho\Delta L$	$\bar{y}\rho\Delta L$
AO	$(0.4)\rho_P$	0	-0.2	0	$-0.08\rho_P$
OB	$\rho_P\sqrt{(0.4)^2+(0.3)^2} = 0.5\rho_P$	0.15	0.2	0.075	$0.1\rho_P$
BC	$\pi(0.4)\rho_Q = 1.257\rho_Q = 0.314\rho_P$	$0.3+0.4 = 0.7$	$0.4 - 2(0.4)/\pi = 0.1453$	$0.88\rho_Q = 0.22\rho_P$	$0.1826\rho_Q = 0.04565\rho_P$
	$\sum\rho\Delta L = 1.214\rho_P$			$\sum\bar{x}\rho\Delta L = 0.295\rho_P$	$\sum\bar{y}\rho\Delta L = 0.06565\rho_P$

$$\therefore \bar{X} = \frac{\sum\bar{x}\Delta W}{\sum\Delta W} = \frac{\sum\bar{x}\rho\Delta L}{\sum\rho\Delta L} = \frac{0.295\rho_P}{1.214\rho_P} = 0.243 \text{ m}$$

$$\therefore \bar{Y} = \frac{\sum\bar{y}\Delta W}{\sum\Delta W} = \frac{\sum\bar{y}\rho\Delta L}{\sum\rho\Delta L} = \frac{0.06565\rho_P}{1.214\rho_P} = 0.054 \text{ m}$$

▪ Example 13.8

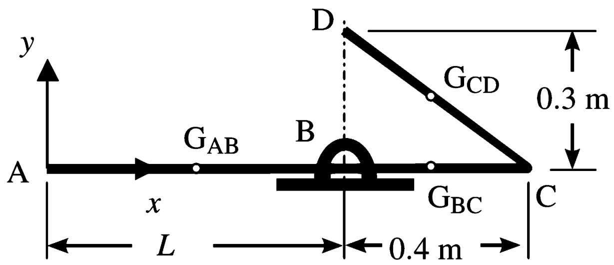
Homogeneous bent rod ABCD is pin jointed at B as shown. Determine the length L to maintain ABC in the horizontal position, and point D is vertically above B.



Solution:

Step 1	Determine the axes. Observe that point G (and C) lies on the line BD. The axes can be placed at any convenient location, like A or B. If A is chosen, then the $\bar{X}$ coordinate is L, if B is chosen, then the $\bar{X}$ coordinate is 0.
Step 2	Segregation into common line segments.
Step 3	Calculate the length (ΔL) of each segments.
Step 4	Determine the $(\bar{x}, \bar{y})$ coordinates of each segment.
Step 5	Construct a table.
Step 6	Solve the coordinate of G for the composite body using $\bar{X} = \frac{\sum\bar{x}\Delta L}{\sum\Delta L}$ and $\bar{Y} = \frac{\sum\bar{y}\Delta L}{\sum\Delta L}$.

Taking axes at A



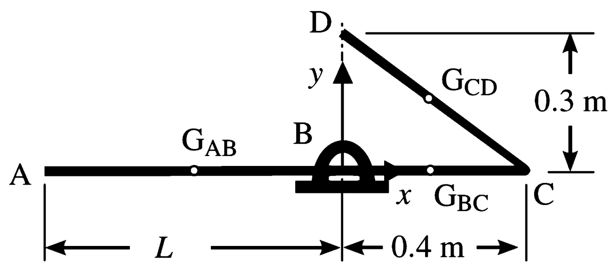
Part	ΔL(m)	$\bar{x}$(m)	$\bar{x}\Delta L(\mathrm{m}^2)$
AB	L	$L/2$	$0.5L^2$
BC	0.4	$L+0.2$	$0.4L+0.08$
CD	$\sqrt{0.4^2+0.3^2}$	$L+0.2$	$0.5L+0.1$
	$\sum\Delta L = L+0.9$		$\sum\bar{x}\Delta L = 0.5L^2+0.9L+0.18$

$$\therefore \bar{X} = \frac{\sum\bar{x}\Delta W}{\sum\Delta W} = \frac{\sum\bar{x}\Delta L}{\sum\Delta L} = \frac{0.5L^2+0.9L+0.18}{L+0.9} = L$$

$$0.5L^2+0.9L+0.18 = L(L+0.9)$$
$$0.5L^2-0.18 = 0$$

thus $L = \pm 0.6$ m $\therefore L = 0.6$ m

Taking axes at B



Part	ΔL(m)	$\bar{x}$(m)	$\bar{x}\Delta L$(m^2)
AB	L	$-L/2$	$-0.5L^2$
BC	0.4	0.2	0.08
CD	$\sqrt{0.4^2+0.3^2}$	0.2	0.1
	$\sum\Delta L = L+0.9$		$\sum\bar{x}\Delta L = -0.5L^2+0.18$

$$\therefore \bar{X} = \frac{\sum\bar{x}\Delta W}{\sum\Delta W} = \frac{\sum\bar{x}\Delta L}{\sum\Delta L} = \frac{-0.5L^2+0.18}{L+0.9} = 0$$

$$-0.5L^2+0.18 = 0$$

thus $L = \pm 0.6$ m $\therefore L = 0.6$ m

13.2.3 Centroid/center of gravity for general shape areas

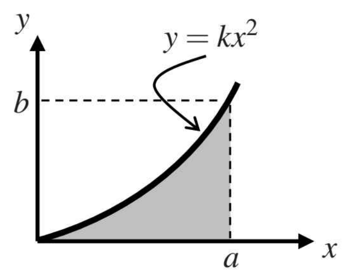
Consider a homogeneous plate with uniform thickness on x-y plane having general shapes. Since the plate has homogeneous thickness, only x and y axes need to be considered. The centroid and center of gravity of this plate are located on the same point.

They are two method of solutions:

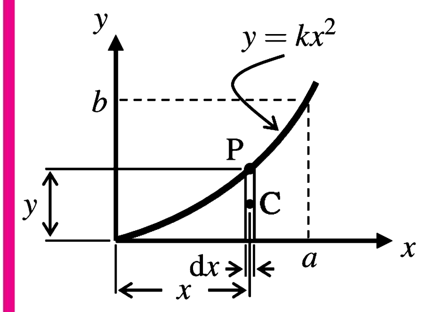
- Vertical element method
- Horizontal element method

The solution strategy is best illustrated by the following examples.

Example 13.9
Determine the centroid of the shaded area by integration.



(a) Solution using the vertical element method



Step 1 Choose a point on the curve, P. Determine its coordinate, $\mathrm{P}(x,y)$.

Step 2 For vertical element method draw an element with thickness dx from P to the x-axis.

Step 3 Form equation of the curve (if applicable), y in terms of x. for the problem above, given that $y = kx^2$
when $y = b$, $x = a$;

$$\therefore k = \frac{b}{a^2}$$

hence, the equation is

$$y = \frac{b}{a^2}x^2$$

solve for area of the shaded region A using $A = \int \mathrm{d}A$ with $\mathrm{d}A = y\mathrm{d}x$ gives

$$\therefore A = \int_0^a \frac{b}{a^2}x^2\mathrm{d}x = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a = \frac{ab}{3}$$

Observe that for the vertical element method, the limits for integration are about the x-axis.

Step 4 The centroid of the element is given by $\bar{x}_{\mathrm{el}} = x$ and $\bar{y}_{\mathrm{el}} = \frac{y}{2}$.
Therefore solving for $\int \bar{x}_{\mathrm{el}}\mathrm{d}A$ gives

$$\int \bar{x}_{\mathrm{el}}\mathrm{d}A = \int_0^a xy\mathrm{d}x = \int_0^a x\frac{b}{a^2}x^2\mathrm{d}x = \int_0^a \frac{b}{a^2}x^3\mathrm{d}x = \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{ba^2}{4}$$

and solving for $\int \bar{y}_{\mathrm{el}}\mathrm{d}A$ gives

$$\int \bar{y}_{\mathrm{el}}\mathrm{d}A = \int_0^a \frac{y}{2}y\mathrm{d}x = \int_0^a \frac{b^2}{2}\frac{x^4}{a^4}\mathrm{d}x = \left[\frac{b^2}{a^4}\frac{x^5}{10}\right]_0^a = \frac{b^2 a}{10}$$

Solve the coordinate of centroid $C(\bar{X},\bar{Y})$ as follows;

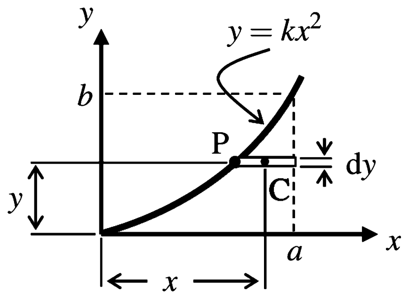
Step 5

$$\bar{X}A = \int \bar{x}_{el}dA \qquad \bar{Y}A = \int \bar{y}_{el}dA$$

$$\bar{X}\frac{ab}{3} = \frac{ba^2}{4} \qquad \bar{Y}\frac{ab}{3} = \frac{ab^2}{10}$$

$$\therefore \bar{X} = \frac{3}{4}a \qquad \therefore \bar{Y} = \frac{3}{10}b$$

(b) Solution using the horizontal element method



Step 1	Choose a point on the curve, P. Determine its coordinate, $P(x,y)$.
Step 2	For vertical element method draw an element with thickness dy from P to the y-axis.

Form equation of the curve (if applicable), x in terms of y. for the problem above, given that $y = kx^2$. When $y = b$, $x = a$, $\therefore k = \frac{b}{a^2}$. Hence, the equation is

$$x = \frac{a}{b^{\frac{1}{2}}}y^{\frac{1}{2}}$$

Step 3 Solving for area of the shaded region A using $A = \int dA$ with $dA = (a-x)dy$ gives

$$\therefore A = \int_0^b a - \frac{ay^{\frac{1}{2}}}{b^{\frac{1}{2}}}dy = \left[ay - \frac{2ay^{\frac{3}{2}}}{b^{\frac{1}{2}}}\right]_0^b = ab - \frac{2ab}{3} = \frac{ab}{3}$$

Observe that for the vertical element method, the limits for integration are about the y-axis.

Step 4

The centroid of the element is given by $\bar{x}_{el} = x + \frac{a-x}{2} = \frac{a+x}{2}$ and $\bar{y}_{el} = y$. Therefore solving for $\int \bar{x}_{el} dA$ gives

$$\int \bar{x}_{el} dA = \int_0^b \frac{a+x}{2}(a-x)dy = \int_0^a \frac{a^2-x^2}{2}dy = \frac{1}{2}\left[a^2 y - \frac{a^2}{b}\frac{y^2}{2}\right]_0^b = \frac{a^2 b}{4}$$

and solving for $\int \bar{y}_{el} dA$ gives

$$\int \bar{y}_{el} dA = \int_0^b y(a-x)dy = \int_0^b y\left(a - \frac{ay^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)dy = \left[\frac{ay^2}{2} - \frac{2ay^{\frac{5}{2}}}{5b^{\frac{1}{2}}}\right]_0^b = \frac{ab^2}{10}$$

Step 5

Solve the coordinate of centroid $C(\bar{X}, \bar{Y})$ as follows;

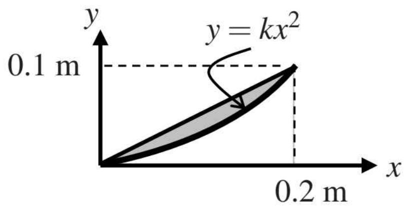
$$\bar{X}A = \int \bar{x}_{el} dA \qquad \bar{Y}A = \int \bar{y}_{el} dA$$

$$\bar{X}\frac{b}{3a} = \frac{a^2 b}{4} \qquad \bar{Y}\frac{b}{3a} = \frac{ab^2}{10}$$

$$\therefore \bar{X} = \frac{3}{4}a \qquad \therefore \bar{Y} = \frac{3}{10}b$$

■ Example 13.10

Determine the centroid of the shaded area by integration.

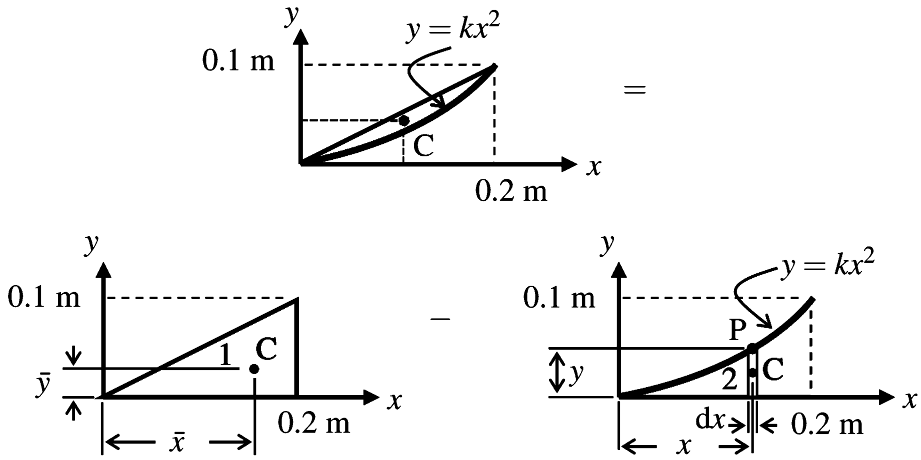


Solution:

Step 1 Determine the axes.

Separate into common components.

Step 2



Step 3 Determine the area (A) and the coordinates of the controid $(\bar{x}, \bar{y})$ of each component.

<u>Component 1:</u>

Area:

$$A = \frac{1}{2}(0.1)(0.2) = 0.01 \text{ m}^2$$

Centroid:

$$\bar{x} = \frac{2}{3}(0.2) = 0.133 \text{ m} \qquad \bar{y} = \frac{1}{3}(0.1) = 0.033 \text{ m}$$

<u>Component 2:</u> (using vertical element method)

Step (a)	Choose a point on the curve, P and determine its coordinate P(x, y).
Step (b)	For vertical element method, draw an element with thickness dx from P to the x-axis

Step (c)

Form equation of the curve (if applicable), y in terms of x.
Given that $y = kx^2$; when $y = 0.1, \quad x = 0.2$

$$\therefore k = \frac{0.1}{0.2^2} \quad \text{hence the equation is } y = 2.5x^2$$

Solving for area A using $A = \int dA$ with $dA = y dx$ gives

$$\therefore A = \int_{0.2}^{0} 2.5x^2 dx = \left[\frac{2.5x^3}{3}\right]_0^{0.2} = \frac{2.5(0.2)^3}{3} = 0.00667 \text{ m}^2$$

Step (d)

Calculate the coordinate of the centroid of the element

$$\bar{x}_{el} = x \qquad \text{and} \qquad \bar{y}_{el} = \frac{y}{2}$$

Solving for $\int \bar{x}_{el} dA$ gives

$$\int \bar{x}_{el} dA = \int_0^{0.2} xy dx = \int_0^{0.2} x2.5x^2 dx = \int_0^{0.2} 2.5x^3 dx$$
$$= \left[\frac{2.5x^4}{4}\right]_0^{0.2} = \left[\frac{2.5(0.2)^4}{4}\right] = 0.001 \text{ m}^3$$

Solving for $\int \bar{y}_{el} dA$ gives

$$\int \bar{y}_{el} dA = \int_0^{0.2} \frac{y}{2} y dx = \int_0^{0.2} 2.5^2 x^4 dx$$
$$= \left[2.5^2 \frac{x^5}{10}\right]_0^{0.2} = \left[2.5^2 \frac{0.2^5}{10}\right] = 0.0002 \text{ m}^3$$

Step (e)

Calculate the coordinate of the centroid of the element using the formulas

$$\bar{X}A = \int \bar{x}_{el} dA \qquad \bar{Y}A = \int \bar{y}_{el} dA$$
$$\bar{X}(0.00667) = 0.001 \qquad \bar{Y}(0.00667) = 0.0002$$
$$\therefore \bar{X} = 0.15 \text{ m} \qquad \therefore \bar{Y} = 0.03 \text{ m}$$

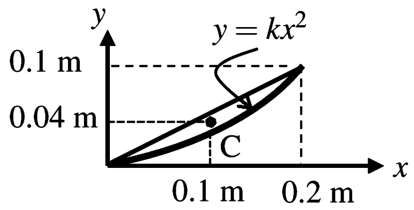
Step 4 Construct a table as follows;

Component	$\Delta A(\mathrm{m}^2)$	$\bar{x}(m)$	$\bar{y}(\mathrm{m})$	$\bar{x}\Delta A(\mathrm{m}^3)$	$\bar{y}\Delta A(\mathrm{m}^3)$
1	0.01	0.133	0.033	0.00133	0.00033
2	−0.00667	0.15	0.03	−0.001	−0.0002
	$\sum \Delta A =$ 0.00333			$\sum \bar{x}\Delta A =$ 0.00033	$\sum \bar{y}\Delta A =$ 0.000133

Step 5 Solve the coordinate of centroid $C(\bar{X}, \bar{Y})$ as follows;

$$\bar{X} = \frac{\sum \bar{x}\Delta A}{\sum \Delta A} = \frac{0.00033}{0.00333} = 0.1 \text{ m} \qquad \bar{Y} = \frac{\sum \bar{y}\Delta A}{\sum \Delta A} = \frac{0.00133}{0.00333} = 0.04 \text{ m}$$

The figure below shows the location of centroid;



13.3 Theorem of Pappus

The theorems was formulated during the third century A.D by Pappus, a Greek geometer for calculating surface area and volume of revolution. Surface area of revolution is the area produced resulting from revolving a line about an axis. For example, the slanting line AB in Figure 13.1(a) rotated about the x-axis will generate a hollow (an ice-cream) cone while the bent line ABC in Figure 13.1(b) will generate a cone with a base.

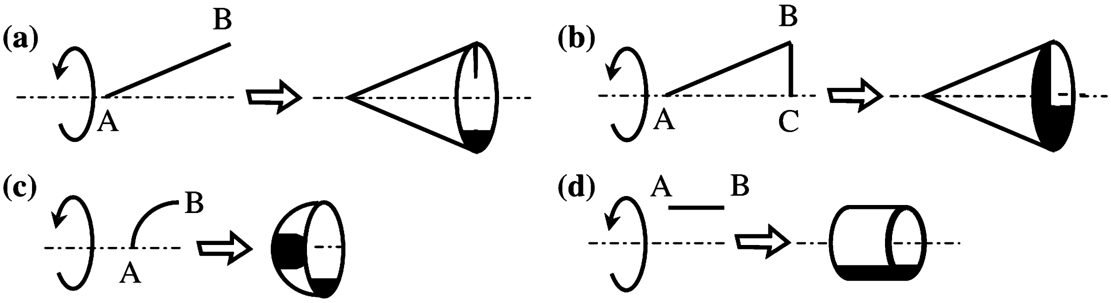


Figure 13.1: Surface area of revolution.

The volume of revolution is the volume which results by generating an area about an axis. For example, a triangular area (shown in Figure 13.2(a)) rotated about the x-axis will generate the volume of a cone. Figure 13.2(b) shows the area of a rectangle producing a cylinder.

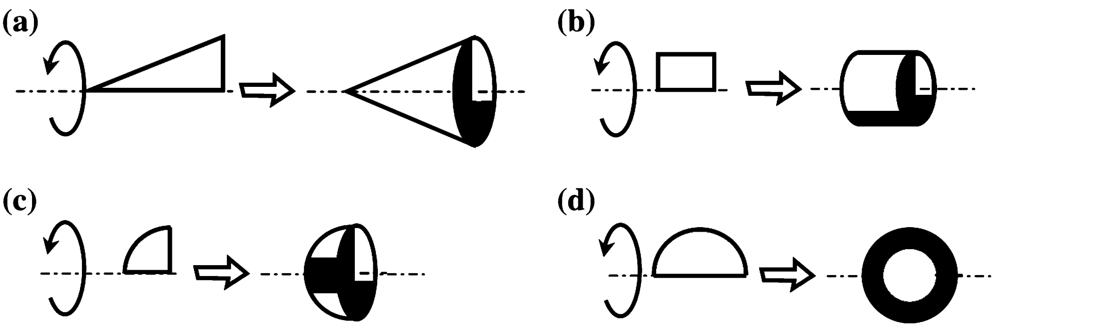


Figure 13.2: Volume of revolution.

It is very important to understand the two processes, i.e. the revolution of a slanting line about an axis will generate the surface area of a cone, and to get the surface area of a cone, we have to rotate a slanting line about an axis. Similarly, a triangular area about an axis will generate the volume of a cone, and to get the volume of a cone, we have to rotate a triangular area about an axis.

Concept 13.1 — Pappus's Theorem 1: Surface area of revolution.
The surface area generated is equal to the total length of the line times the distance covered by the centroid of that line while generating the surface.

For a line rotated about the x-axis, surface area of revolution is given by;

$$A = 2\pi\bar{Y}\sum \Delta L = 2\pi\sum \bar{y}\Delta L$$

For a line rotated about the y-axis, surface area of revolution is given by;

$$A = 2\pi\bar{X}\sum \Delta L = 2\pi\sum \bar{x}\Delta L$$

Concept 13.2 — Pappus's Theorem 2: Volume of revolution.
The volume of revolution is equal to the total area times the distance covered by centroid of the area while generating the volume.

For an area rotated about the x-axis, volume of revolution is given by;

$$V = 2\pi\bar{Y}\sum \Delta A = 2\pi\sum \bar{y}\Delta A$$

For an area rotated about the y-axis, volume of revolution is given by;

$$V = 2\pi\bar{X}\sum \Delta A = 2\pi\sum \bar{x}\Delta A$$

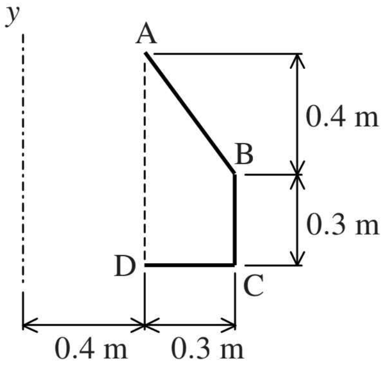
(!) The line/area involved must be located on one side of the axis of rotation only!

■ Example 13.11

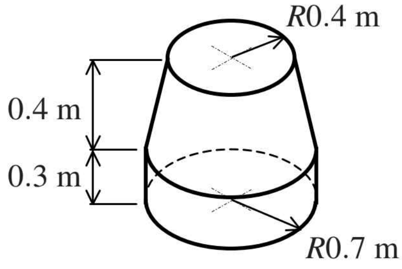
Determine the surface area generated if the composite line is rotated 360° about the y-axis (Figure (a)).

or

Determine the surface area of the hollow component of negligible thickness using Theorem of Pappus (Figure (b)).



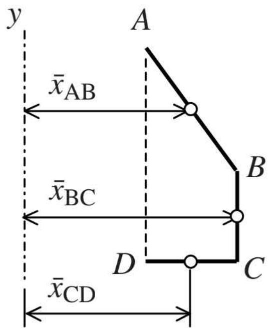
(a)



(b)

Solution:
Both problem have the same solution. The strategy to solve the problem is the same as the procedure for centroid/center of gravity.

- Separate the composite line into common segments.
- Determine the length of each segment.
- Determine the x coordinate of the centroid (rotated about y-axis).
- Form a table to solve the problem.



Line segment	ΔL(m)	$\bar{x}$(m)	$\bar{x}\Delta L$(m^2)
AB	$\sqrt{0.3^2+0.4^2}=0.5$	$0.4+\frac{0.3}{2}=0.55$	0.275
BC	0.3	0.4 + 0.3 = 0.7	0.21
CD	0.3	$0.4+\frac{0.3}{2}=0.55$	0.165
	$\sum\Delta L=1.1$		$\sum\bar{x}\Delta L=0.65$

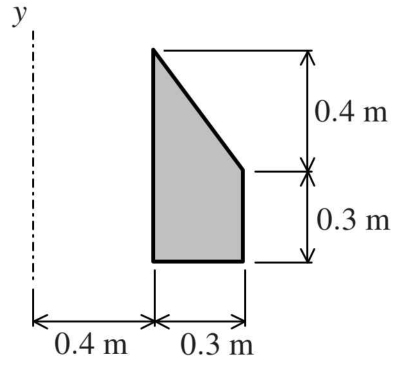
Surface area of revolution, $A=2\pi\bar{X}\sum\Delta L=2\pi\sum\bar{x}\Delta L=2\pi(0.65)=4.084\ \text{m}^2$

■ Example 13.12

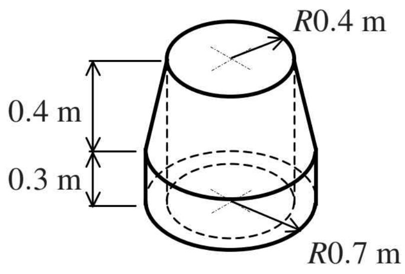
Determine the volume generated if the shaded area is rotated 360° about the y-axis (Figure (a)).

or

Determine the volume of the component using Theorem of Pappus (Figure (b)).



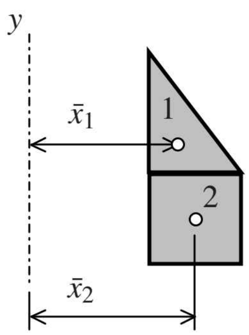
(a)



(b)

Solution:
Both problem have the same solution. The strategy to solve the problem is the same as the procedure for centroid/center of gravity.

- Separate the shaded area into common segment.
- Determine the area of each component.
- Determine the x coordinate of the centroid (rotated about y-axis).
- Form a table to solve the problem.

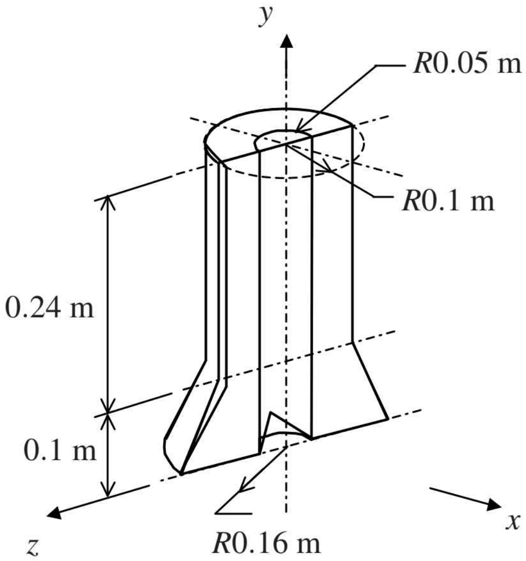


Area segment	$\Delta A(\mathrm{m}^2)$	$\bar{x}(\mathrm{m})$	$\bar{x}\Delta A(\mathrm{m}^3)$
1	$\frac{1}{2}(0.3)(0.4) = 0.06$	$0.4 + \frac{1}{3}(0.3) = 0.5$	0.03
2	$(0.3)(0.3) = 0.09$	$0.4 + \frac{0.3}{2} = 0.55$	0.495
	$\sum \Delta A = 0.15$		$\sum \bar{x}\Delta A = 0.795$

Volume of revolution, $V = 2\pi\bar{X}\sum\Delta A = 2\pi\sum\bar{x}\Delta A = 2\pi(0.795) = 4.995\ \mathrm{m}^3$

▪ Example 13.13

Determine the volume and surface area of the component shown using Theorem of Pappus.

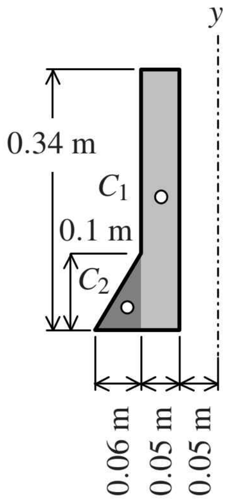


Solution:

Volume of the component

Volume is generated by rotating the area 180° about the y-axis. Determine the z coordinate of the centroid.

- Separate half of the cross sectional area into common area segment.
- Determine the area of each segment.
- Determine the z coordinate of the centroid (rotated about y-axis).
- Form a table to solve the problem.



Area segment	$\Delta A(\mathrm{m}^2)$	$\bar{z}(\mathrm{m})$	$\bar{z}\Delta A(\mathrm{m}^3)$
1	$0.05(0.34) = 0.017$	$0.05 + 0.025 = 0.075$	0.001275
2	$\frac{1}{2}(0.06)(0.1) = 0.003$	$0.1 + \frac{1}{3}(0.06) = 0.12$	0.00036
	$\sum \Delta A = 0.02$		$\sum \bar{z}\Delta A = 0.001635$

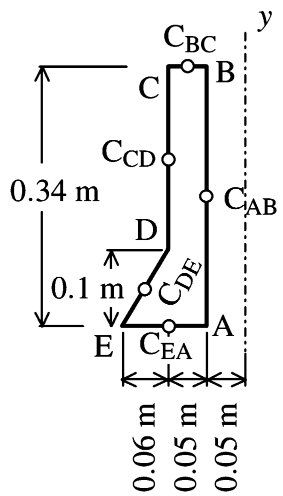
Centroid is given as $\bar{Z} = \frac{\sum \bar{z}\Delta A}{\sum \Delta A}$. From Theorem of Pappus, the volume of revolution is given as $V = 2\pi \bar{Z} \sum \Delta A$. Since to get the volume involves half of a revolution (180°), the equation becomes

$$V = \frac{1}{2}(2\pi \bar{Z} \sum \Delta A) = \frac{1}{2}(2\pi \sum \bar{z}\Delta A) = \pi(0.001635) = 0.005137 \text{ m}^3$$

Surface area of the component

Surface area of the component is generated by rotating the lines 180° about the y-axis. Determine the centroid about line the z-axis first.

- Separate half of the cross sectional perimeter into common line segment.
- Determine the length of each segment.
- Determine the z coordinate of the centroid (rotated about y-axis).
- Form a table to solve the problem.

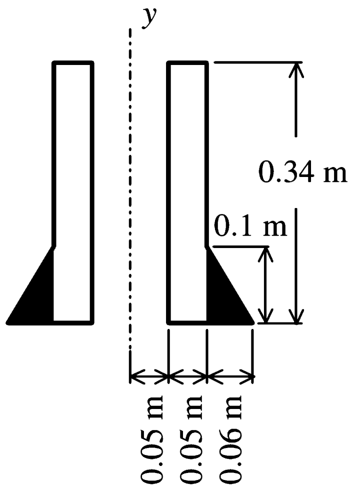


Line segment	ΔL(m^2)	$\bar{z}$(m)	$\bar{z}\Delta L$(m^2)
AB	0.34	0.05	0.017
BC	0.05	0.075	0.00375
CD	0.24	0.1	0.024
DE	$\sqrt{0.06^2 + 0.1^2} = 0.117$	0.13	0.01521
EA	0.11	0.105	0.01155
	$\sum \Delta L = 0.857$		$\sum \bar{z}\Delta L = 0.0715$

Centroid is given as $\bar{Z} = \frac{\sum \bar{z}\Delta L}{\sum \Delta L}$. From Theorem of Pappus, the surface area of revolution is given as

$$A = \frac{2\pi \bar{Z} \sum \Delta L}{2} = \frac{2\pi \sum \bar{z}\Delta L}{2} = \frac{2\pi(0.0715)}{2} = 0.2246 \text{ m}^2$$

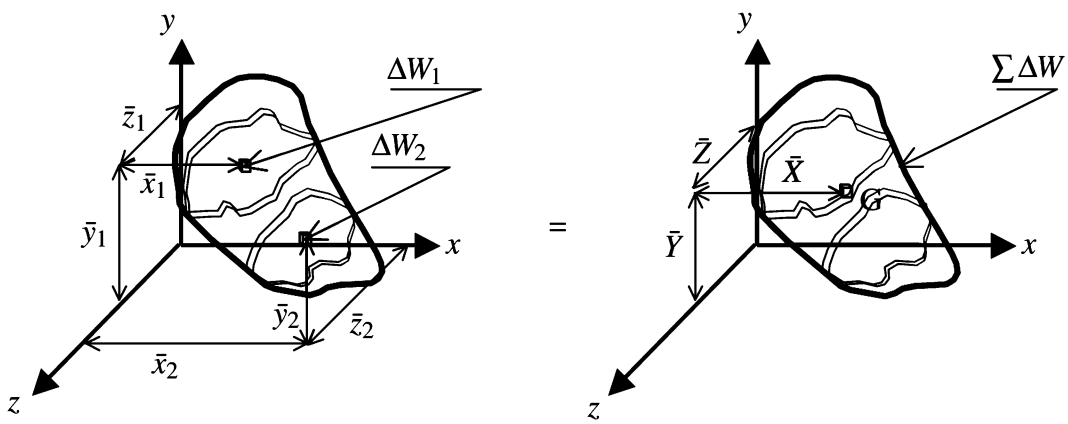
However, the solution is still incomplete as there are two more areas that need to be considered i.e the cross sectional area of the component.



$$\text{cross sectional area} = 2[(0.05)(0.34) + \frac{1}{2}(0.1)(0.06)]$$
$$= 0.04 \text{ m}^2$$
$$\therefore \text{total surface area} = 0.2246 + 0.04$$
$$= 0.2646 \text{ m}^2$$

13.4 Centroid and centre of gravity of a volume (3D)

The same approach applied in the 2D analysis could be extended to determine the centroid and centre of gravity of 3D composite bodies.



$$M_y = \text{moment of distributed forces} = \text{moment of the resultant}$$
$$= \bar{x}_1 \Delta W_1 + \bar{x}_2 \Delta W_2 + \bar{x}_3 \Delta W_3 + \cdots = \bar{X} \sum \Delta W$$
$$= \sum \bar{x} \Delta W = \bar{X} \sum \Delta W$$

Therefore the coordinates of the center of gravity $G(\bar{X}, \bar{Y}, \bar{Z})$ about the x-axis, y-axis and z-axis of a 3D composite body are given by

$$\bar{X} = \frac{\sum \bar{x} \Delta W}{\sum \Delta W}$$

$$\bar{Y} = \frac{\sum \bar{y} \Delta W}{\sum \Delta W}$$

$$\bar{Z} = \frac{\sum \bar{z} \Delta W}{\sum \Delta W}$$

where;
$\sum \Delta W$ = weight of body
ΔW = weight of element
$\bar{x}$ = coordinate of c.g. of element along x-axis
$\bar{y}$ = coordinate of c.g. of element along y-axis
$\bar{z}$ = coordinate of c.g. of element along z-axis
$\bar{X}$ = coordinate of c.g. of body along x-axis
$\bar{Y}$ = coordinate of c.g. of body along y-axis
$\bar{Z}$ = coordinate of c.g. of body along z-axis

with

	where;
$W = \rho g V$	ρ = density of the body
	g = 9.81 m/s^s
$\therefore \Delta W = \rho g \Delta V$	V = volume of the body
	ΔV = volume of element

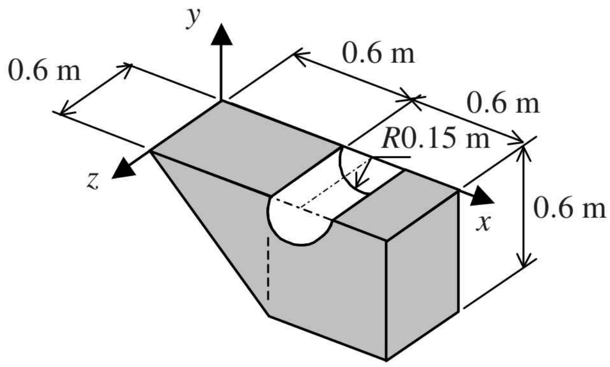
For bodies with constant values of ρ and g, the equations for the coordinates of the center of gravity G$(\bar{X}, \bar{Y}, \bar{Z})$ can be simplified further as

$$\bar{X} = \frac{\sum \bar{x}\Delta V}{\sum \Delta V} \qquad \bar{Y} = \frac{\sum \bar{y}\Delta V}{\sum \Delta V} \qquad \bar{Z} = \frac{\sum \bar{z}\Delta V}{\sum \Delta V}$$

The analysis strategy is shown in the following example.

■ Example 13.14

Determine the centroid of the body shown.



Solution:

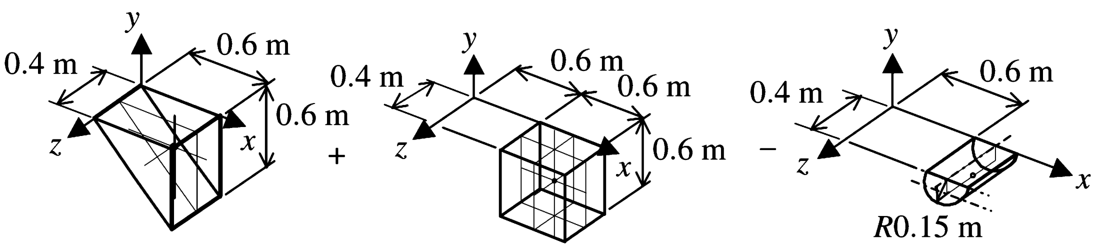
Step 1	Determine the axes.
Step 2	Segregate the body into common components. Ensure that the combination of the segregated components would produce the original body. Volumes of holes are subtracted.
Step 3	Calculate the volume, ΔV of each component.
Step 4	Determine the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of each component.
Step 5	Construct a table.

Step 6

Since ρ and g are constant, the c.g. coordinates are given by;

$$\bar{X} = \frac{\sum \bar{x}\Delta V}{\sum \Delta V} \qquad \bar{Y} = \frac{\sum \bar{y}\Delta V}{\sum \Delta V} \qquad \bar{Z} = \frac{\sum \bar{z}\Delta V}{\sum \Delta V}$$

Analysis is done about the x and y axes only as the component is symmetrical about the z-axis. Observation shows that the location of centroid about the z-axis, $\bar{Z} = 0.3$ m.

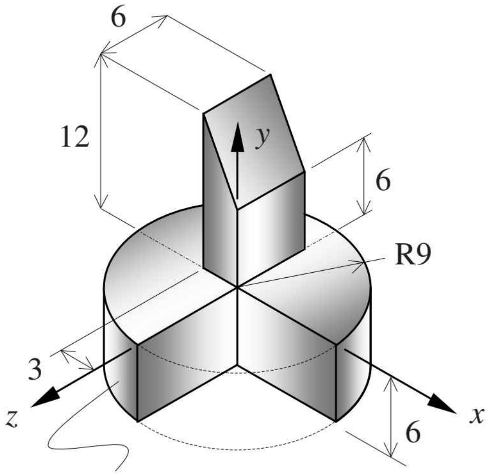


Part	ΔV(m)	$\bar{x}$(m)	$\bar{y}$(m)	$\bar{x}\Delta V$(m^2)	$\bar{y}\Delta V$(m^2)
1	$\frac{1}{2}(0.4)(0.6)(0.6) = 0.228$	$\frac{2}{3}(0.6) = 0.4$	$-\frac{1}{3}(0.6) = -0.2$	0.1152	-0.0576
2	$(0.4)(0.6)(0.6) = 0.144$	$0.6 + 0.3 = 0.9$	-0.3	0.1296	-0.0432
3	$\pi(0.15)^2(0.4) = -0.0283$	$0.6 + 0.15 = 0.75$	$\frac{-4(0.15)}{3\pi} = -0.0637$	-0.0212	0.0018
	$\sum \Delta V = 0.4037$			$\sum \bar{x}\Delta V = 0.2236$	$\sum \bar{y}\Delta V = -0.099$

$$\therefore \bar{X} = \frac{\sum \bar{x}\Delta V}{\sum \Delta V} = \frac{0.2236}{0.4037} = 0.554 \text{ m} \qquad \therefore \bar{Y} = \frac{\sum \bar{y}\Delta V}{\sum \Delta V} = \frac{-0.099}{0.4037} = -0.245 \text{ m}$$

▪ Example 13.15

Determine the centroid of the machine component shown.



Three-quarter circle

All dimensions in mm

Solution:

Machine component = Full circle - quarter circle + rectangular + triangle

Part	ΔV (mm)	$\bar{x}$ (mm)	$\bar{y}$ (mm)	$\bar{z}$ (mm)	$\bar{x}\Delta V$ (mm^2)	$\bar{y}\Delta V$ (mm^2)	$\bar{z}\Delta V$ (mm^2)
Full circle	$\pi(9)^2(6)$	0	-3	0	0	-4580.4	0
Quarter circle	$-\frac{\pi(9)^2(6)}{4}$	$\frac{4(9)}{3\pi}$	-3	$-\frac{4(9)}{3\pi}$	-1458.1	1145.1	-1458.1
Rectangle	$(6)(6)(3)$	-1.5	3	-3	-162	324	-324
Triangle	$\frac{1}{2}(6)(6)(3)$	$-\frac{2}{3}(3)$	$6+\frac{1}{3}(6)$	-3	-108	432	-162
Sum	1307.1				-1728.1	-2679.3	-1944.1

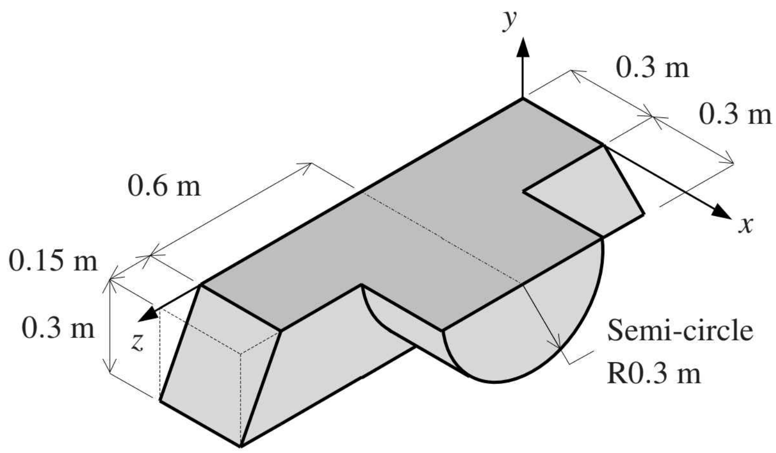
$$\therefore \bar{X} = \frac{\sum \bar{x}\Delta V}{\sum \Delta V} = \frac{-1728.1}{1307.1} = -1.322 \text{ mm}$$

$$\bar{Y} = \frac{\sum \bar{y}\Delta V}{\sum \Delta V} = \frac{2679.3}{1307.1} = -2.05 \text{ mm}$$

$$\bar{Z} = \frac{\sum \bar{z}\Delta V}{\sum \Delta V} = \frac{-1944.1}{1307.1} = -1.487 \text{ mm}$$

■ Example 13.16

Determine the centroid of the symmetrical composite body shown.



Solution:

Part	ΔV (m)	$\bar{x}$ (m)	$\bar{y}$ (m)	$\bar{z}$ (m)	$\bar{x}\Delta V$ (m^2)	$\bar{y}\Delta V$ (m^2)	$\bar{z}\Delta V$ (m^2)
1	$(1.2)(0.3)(0.3)$	0.15	−0.15	0.6	0.0162	−0.0162	0.0648
2	$\frac{1}{2}(0.15)(0.3)(0.3)$	0.15	−0.2	1.25	0.0010125	−0.00135	0.008775
3	$\frac{1}{2}(0.15)(0.3)(0.3)$	0.15	−0.2	−0.05	0.0010125	−0.00135	−0.0003375
4	$\frac{\pi}{2}(0.3)^2(0.3)$	0.45	−0.1273	0.6	0.03817	−0.0108	0.0509
Sum	0.2063				0.0564	−0.0297	0.1241

$$\therefore \bar{X} = \frac{\sum \bar{x}\Delta V}{\sum \Delta V} = \frac{0.0564}{0.206323} = 0.2734 \text{ m}$$

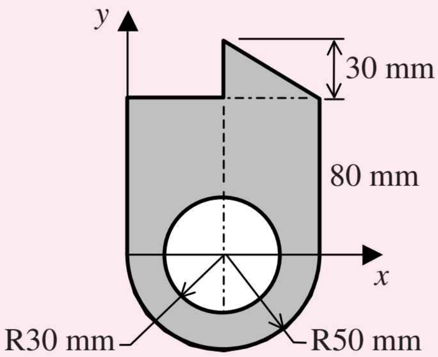
$$\bar{Y} = \frac{\sum \bar{y}\Delta V}{\sum \Delta V} = \frac{-0.0297}{0.206323} = -0.1439 \text{ m}$$

$$\bar{Z} = \frac{\sum \bar{z}\Delta V}{\sum \Delta V} = \frac{0.1241}{0.206323} = 0.6015 \text{ m}$$

13.5 Example questions

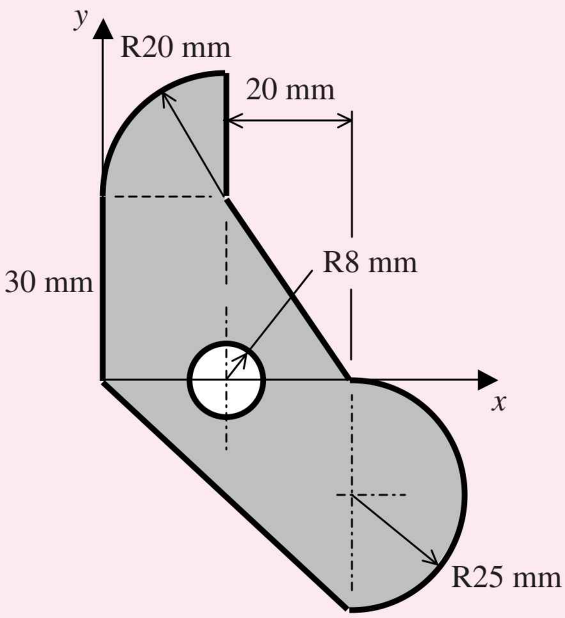
Exercise 13.1

Determine the centroid of the composite body shown.



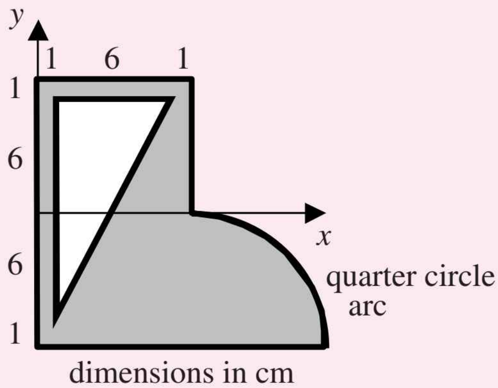
Exercise 13.2

Determine the centroid of the composite body shown.



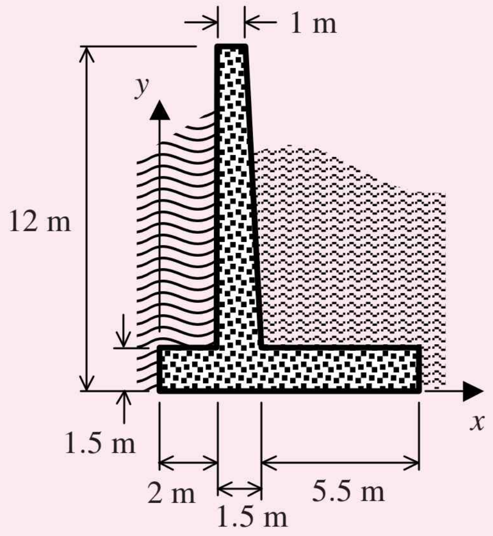
Exercise 13.3

Determine the centroid of the composite body shown.



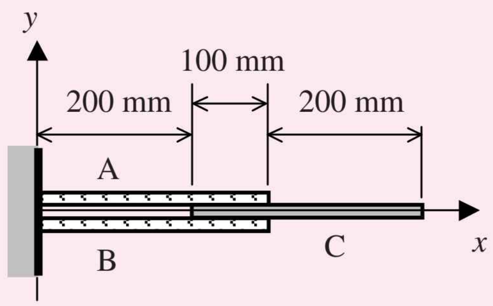
Exercise 13.4

Determine the weight and centre of gravity of the concrete wall of a dam. The wall is 10 m long with a density of 24 kN/m^3.



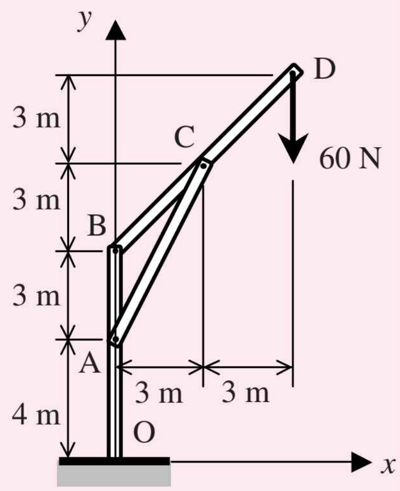
Exercise 13.5

Two metal plates, A and B and an aluminium plate C are sandwich together. The plates are 20 mm thick and 200 mm wide. Determine the centre of gravity of the combined body. Densities of metal and aluminium are 7.85 Mg/m^3 and 2.71 Mg/m^3 respectively.



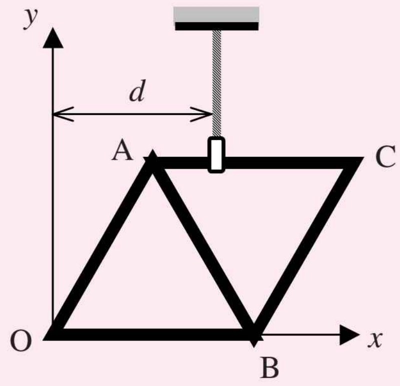
Exercise 13.6

The weight of the three member structure is 4 N/m. Determine location of the centre of gravity. Hence, determine the reaction at O.



Exercise 13.7

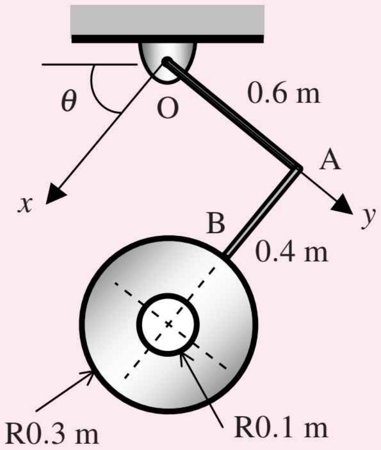
The truss OACB is made up of five 4 m length rods with a mass of 7 kg/m is lifted by a cable. Determine the distance d so that the truss maintains at the position shown.



Exercise 13.8

The figure shows a pendulum made up of a bent rod OAB welded to a hollow plate. Determine the centre of gravity about the x-y axes. Hence, calculate the angle θ if the pendulum is pinned jointed at O.

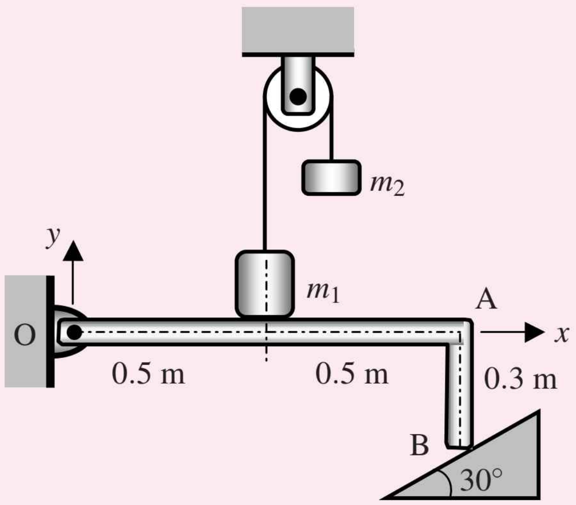
Given:
Mass of rod OAB is 5 kg/m
Mass of the plate material is 20 kg/m^2



Exercise 13.9

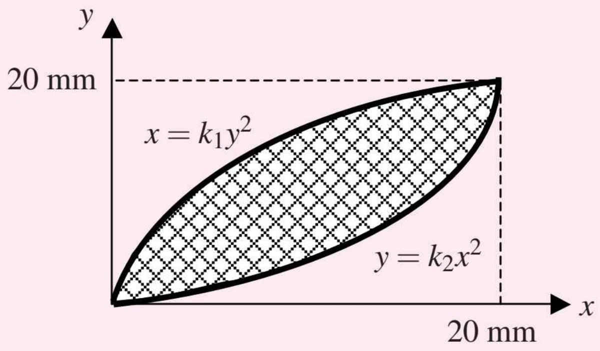
The 130 kg bent rod OAB is supported and loaded as shown. Mass $m_1 = 50$ kg and $m_2 = 35$ kg. The surface at B is smooth. Determine;

a. The centroid of the bent rod.
b. The reactions at O and B.



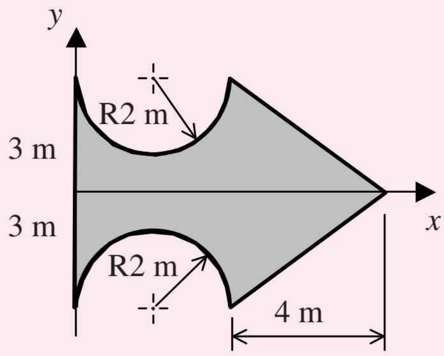
Exercise 13.10

Determine centroid of the shaded area using integration.



Exercise 13.11

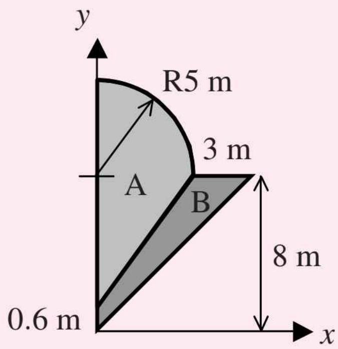
Determine centroid of the shaded area and the volume generated if rotated 90° about the y-axis.



Exercise 13.12

The figure shows cross sectional area of a composite body made up of two different materials. Densities of materials A and B are 10.5×10^3 kg/m3 and 19.3×10^3 kg/m3 respectively. Using Theorem of Pappus, determine;

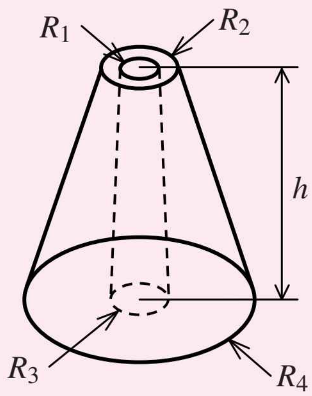
a. Volume of both bodies if rotated 360° about the y-axis.
b. The mass of each body.



Exercise 13.13

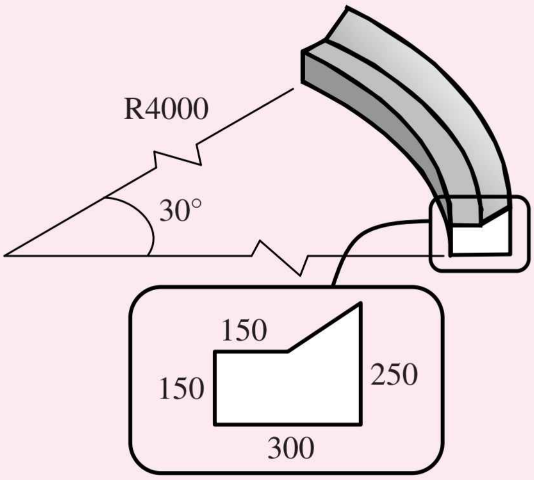
Determine surface area of the composite body.

Given;
$R_1 = 10$ mm
$R_2 = 30$ mm
$R_3 = 20$ mm
$R_4 = 60$ mm
$h = 60$ mm



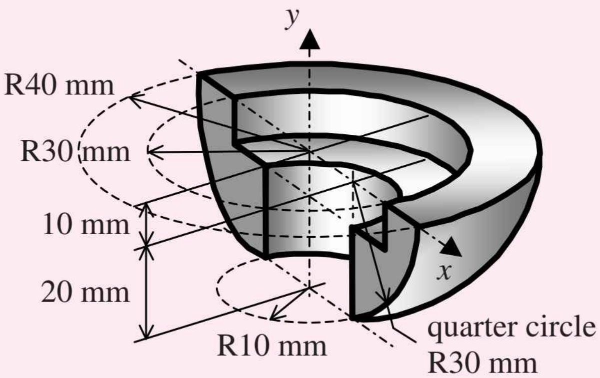
Exercise 13.14

Determine surface area of the component shown using Theorem of Pappus. Dimension shown in mm.



Exercise 13.15

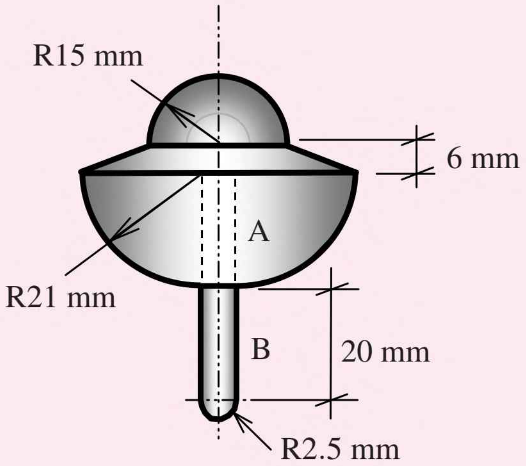
Determine surface area of the component shown using Theorem of Pappus. Dimension shown in mm.



Exercise 13.16

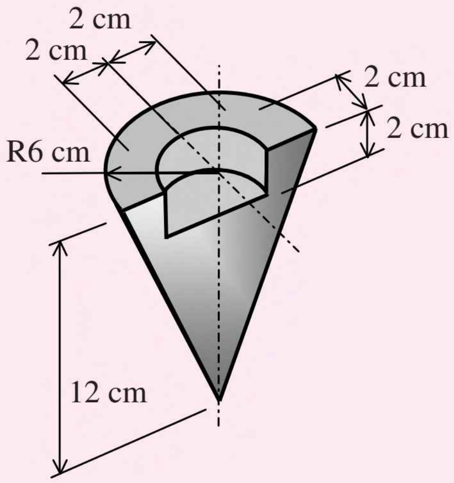
Figure shows a top made up of two different materials. Densities of material A and B are 5 kg/cm^3 and 25 kg/cm^3 respectively. Using the Theorem of Pappus, determine

a. Volumes of A and B.
b. Mass of the top.



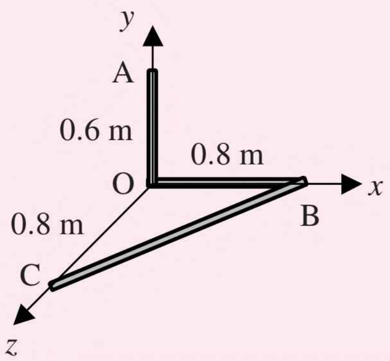
Exercise 13.17

Determine the surface area and volume of the component shown using the Theorem of Pappus.



Exercise 13.18

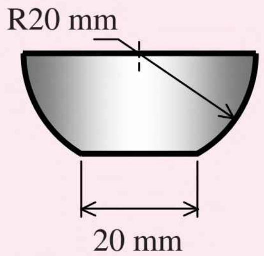
Determine the centre of gravity of the bent rod shown. Rod AO is made of metal while the rod OBC is aluminium. The ratio of their densities is 2.84 : 1.

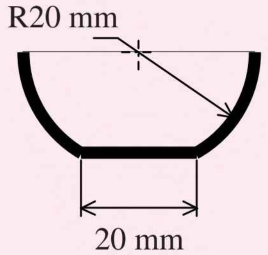


Exercise 13.19

The figure shows the side view of a uniform thickness soup bowl with a flat base.

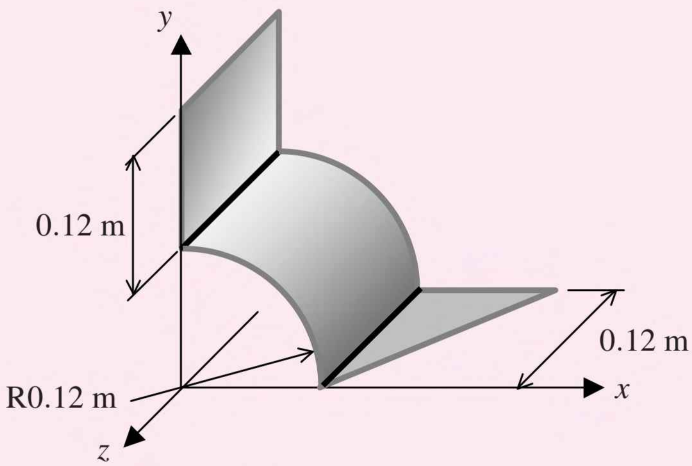
Determine the centroid
a. of the bowl full of soup,
b. of the frame of the bowl.





Using the Theorem of Pappus, determine
c. volume of the bowl,
d. surface area of the bowl.

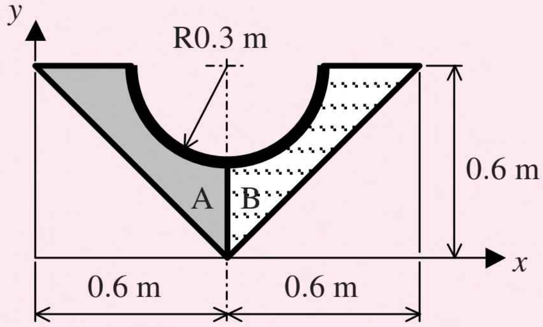
Exercise 13.20
Determine the location of the centre of gravity of the uniform thickness metal sheet shown.



Answer: $\bar{x} = 0.065$ m, $\bar{y} = 0.0975$ m and $\bar{z} = -0.063$ m

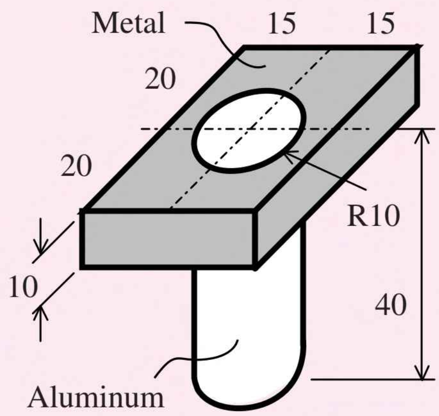
Exercise 13.21

The component shown is made up of two materials. Ratio of the density between the materials A and B is 4:1. Determine the centre of gravity.



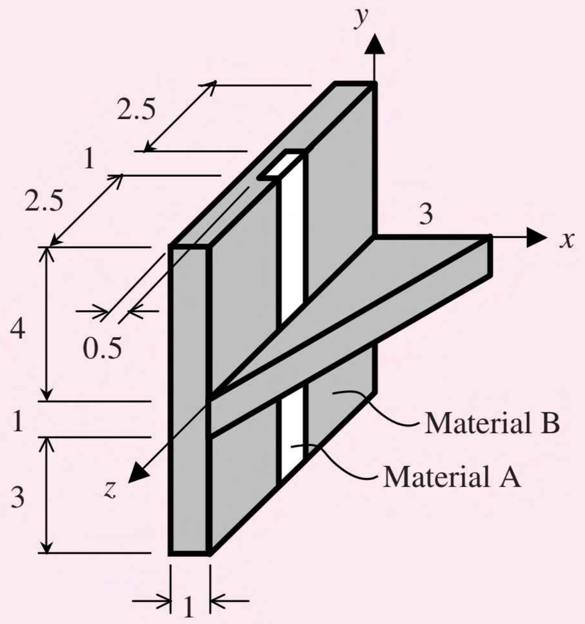
Exercise 13.22

An aluminium cylinder inserted through a metal block is as shown. Determine the centre of gravity if the density of aluminium is 2.8 g/cm^3 and for metal 7.7 g/cm^3. Dimensions in cm.



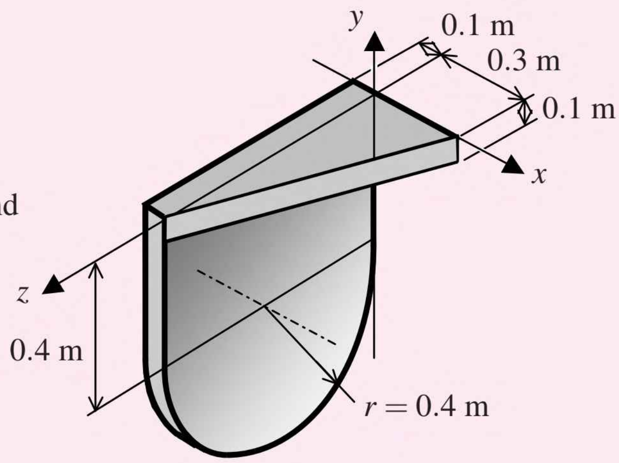
Exercise 13.23

A composite body is made up of two materials. Determine the x coordinate of the centroid and centre of gravity of the composite body. The density of material A is 0.8 the density of material B. Dimensions in cm.



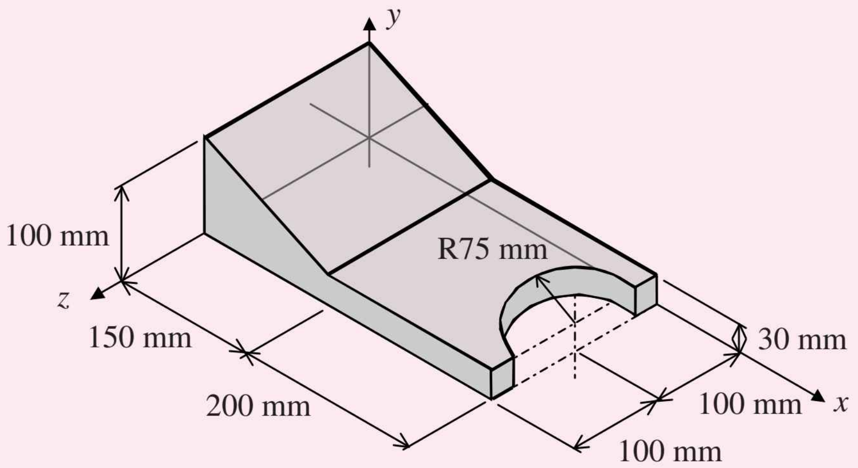
Exercise 13.24

Determine the centroid ($\bar{X}$, $\bar{Y}$ and $\bar{Z}$) of the component shown.

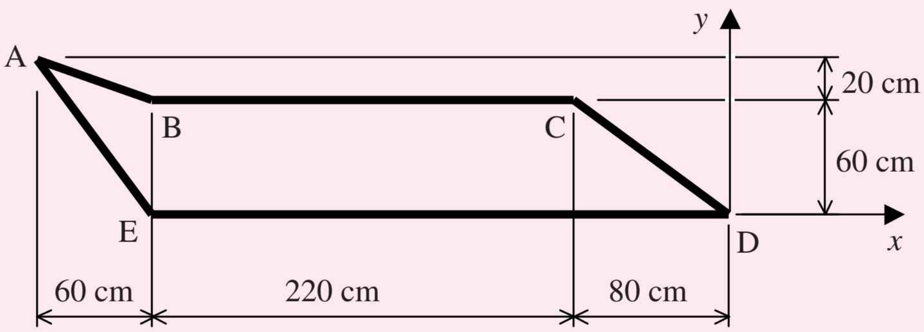


Answer: $\bar{X} = -0.0239$ m, $\bar{Y} = -0.3387$ m and $\bar{Z} = 0.3768$ m

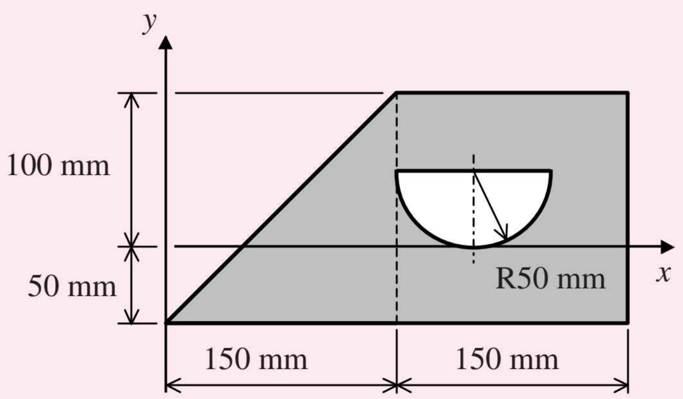
Exercise 13.25
Determine the center of gravity ($\bar{X}$, $\bar{Y}$ and $\bar{Z}$) of the machine element shown.



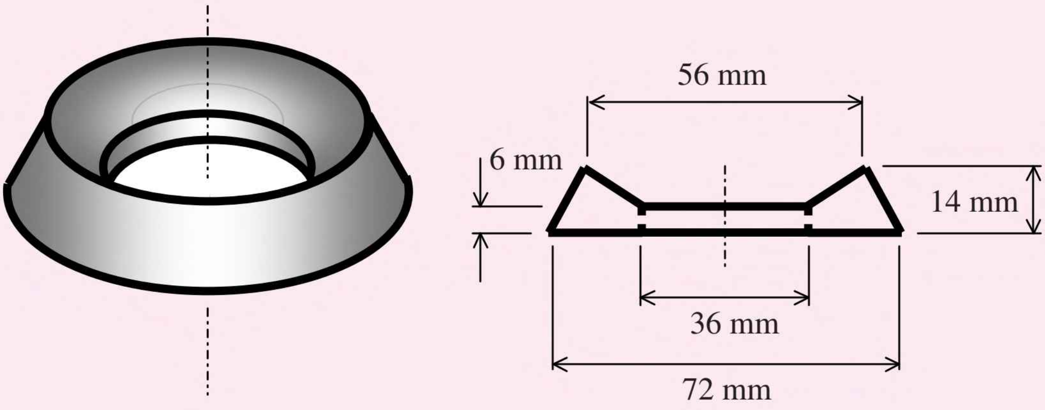
Exercise 13.26
Determine the surface area of the component shown if rotated 180° about the x-axis. Sketch the resulting shape.



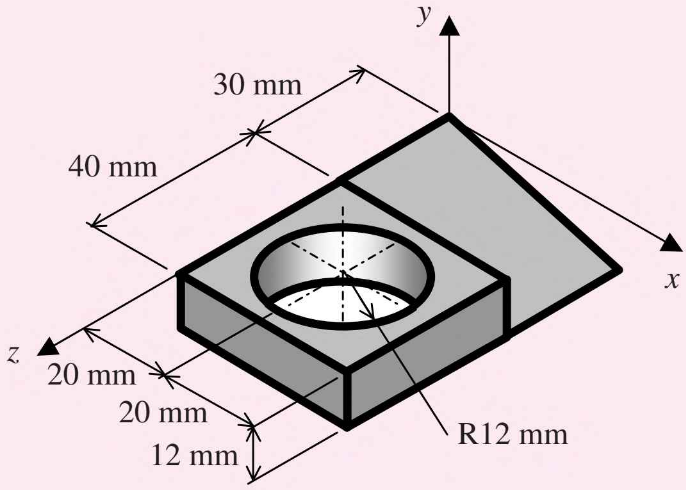
Exercise 13.27
Locate the centroid of the plane area shown. Using the Theorem of Pappus, determine the volume of the solid obtained by rotating the area about the y-axis.



Exercise 13.28
Determine the amount of paint required (in litres) to paint 10,000 components if a litre of paint can cover an area of 10 m^2.

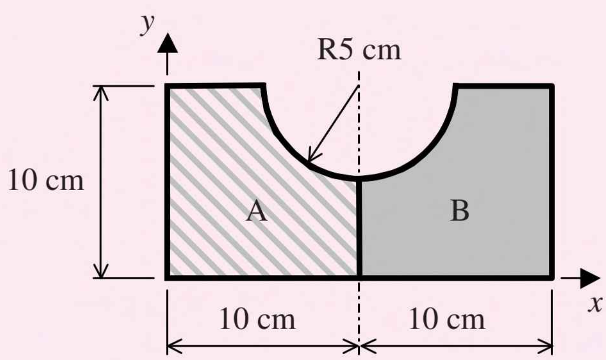


Exercise 13.29
Determine the centroid of the component shown.

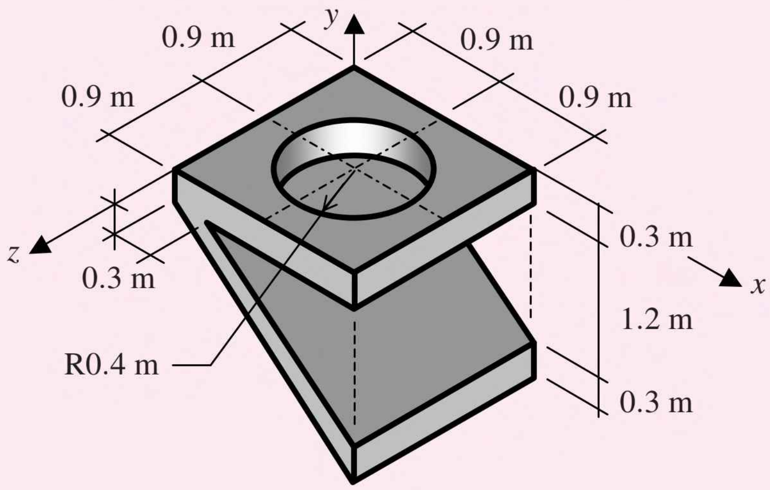


Exercise 13.30

Determine the centre of gravity of the composite body shown. The ratio of the density between materials A and B is 1:4. Determine also the centroid of the area if the plate is made up of the same material.



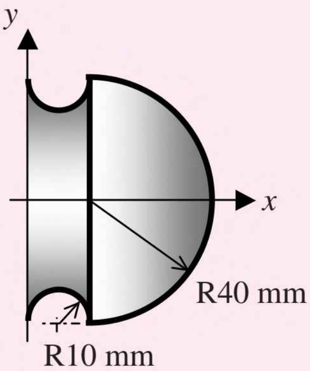
Exercise 13.31
Determine the centroid ($\bar{X}$, $\bar{Y}$ and $\bar{Z}$) for the composite body shown.



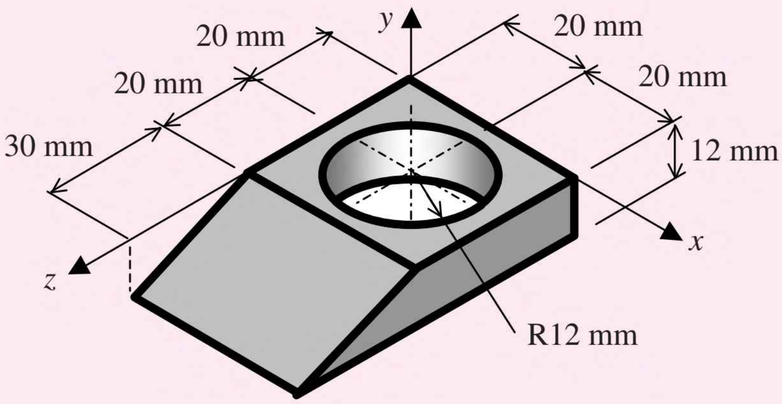
Exercise 13.32

The door knob shown is made of a 5200 kg/m^3 density material. Determine

a. the surface area,
b. the weight of the door knob.

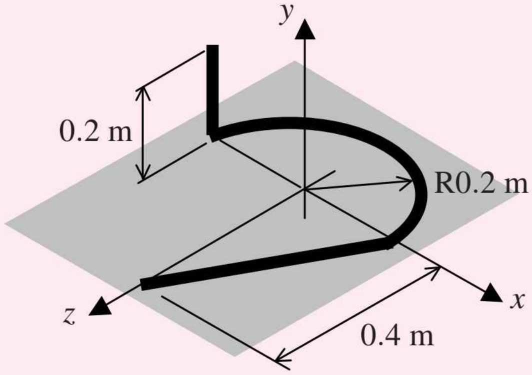


Exercise 13.33
Determine the centroid ($\bar{X}$, $\bar{Y}$ and $\bar{Z}$) for the composite body shown.

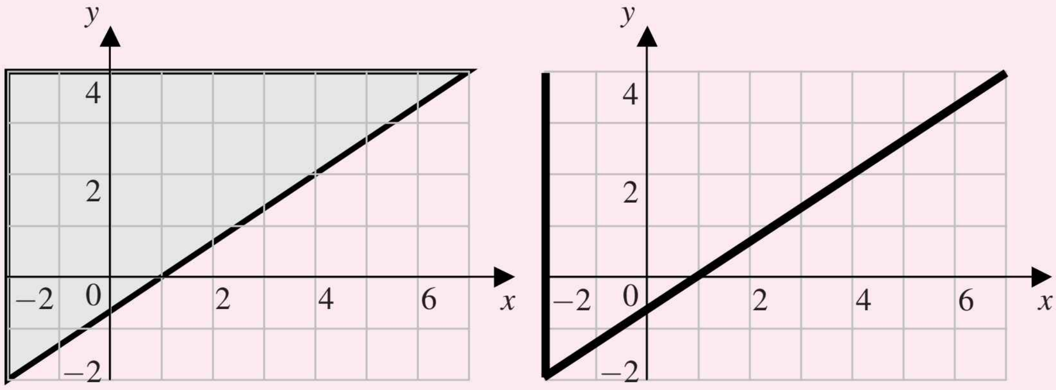


Exercise 13.34

Determine the centroid ($\bar{X}$, $\bar{Y}$ and $\bar{Z}$) for the composite line shown.

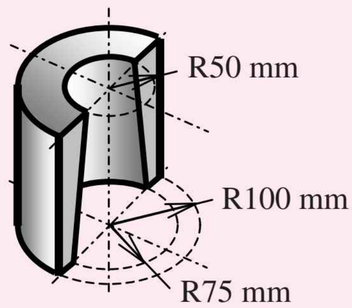


Exercise 13.35
Determine the centroid for both components.



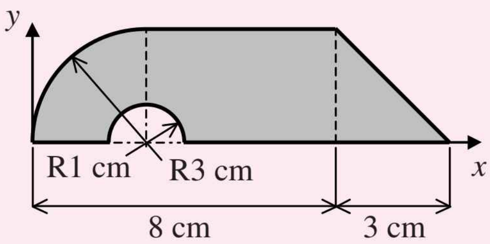
Exercise 13.36

Determine the surface area and volume of revolution of the component shown.



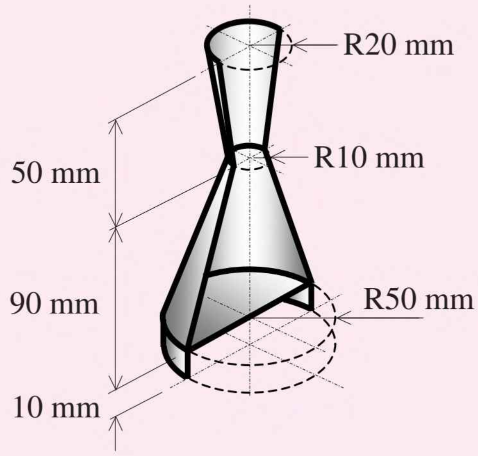
Exercise 13.37

a. Locate the centroid ($\bar{X}$, $\bar{Y}$) of the shaded area shown in the figure.
b. Determine the volume generated if the area is rotated 180° about the x–axis. Sketch the resulting component.



Exercise 13.38

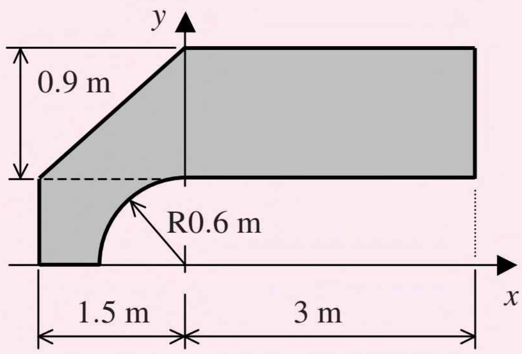
Determine the surface area and volume of revolution of the beaker shown.



Exercise 13.39

The figure shows a machine component. Determine

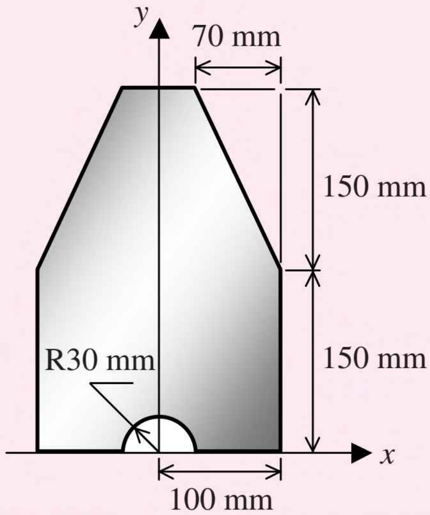
a. centroid of the shaded area,
b. volume generated when the shaded area is rotated 180° about the x–axis.
c. sketch the resulting body in question (b.).



Exercise 13.40

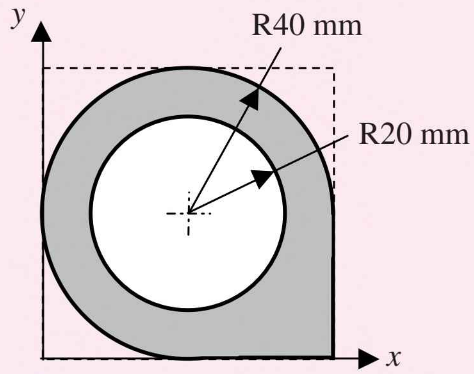
The figure shows a cross section of a machine component which is symmetrical about the y–axis. Determine the location of the centroid refering to the axes, limiting analysis to four (4) components only.

Using Theorem of Pappus, determine the volume generated if the component is revolved 180° about the x–axis and sketch the resulting component (in 3D).



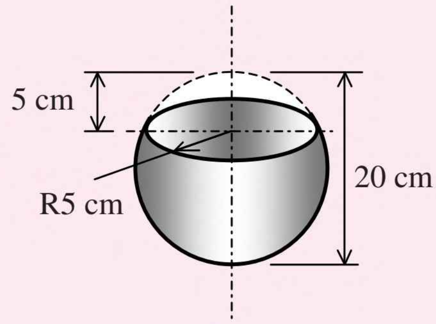
Exercise 13.41

Determine the volume generated if the shaded area shown in the figure is rotated 360° about the y–axis by using the Theorem of Pappus.



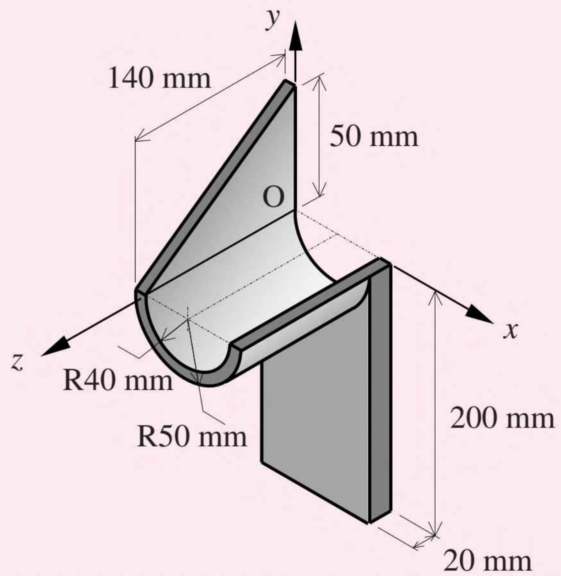
Exercise 13.42

The figure shows a bowl made of a cut–off hollow sphere. Draw the cross–sectional segment (complete with the appropriate dimensions and approximate location of the centroid) needed in order to determine the area of the surface of revolution using the Theorem of Pappus. Do not calculate the area of surface of revolution.



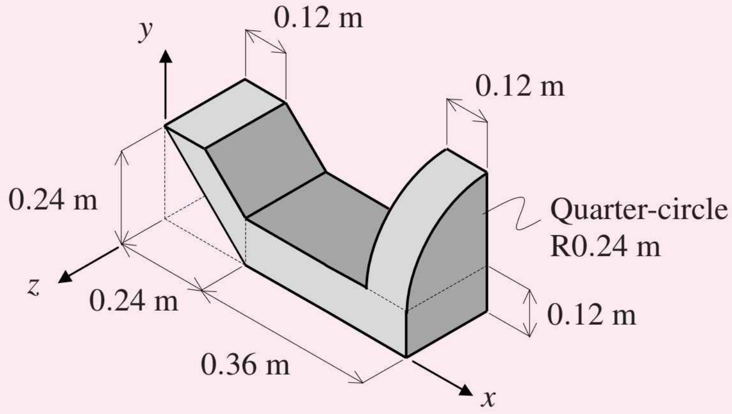
Exercise 13.43

Determine the centroid of the homogeneous composite body.



Exercise 13.44

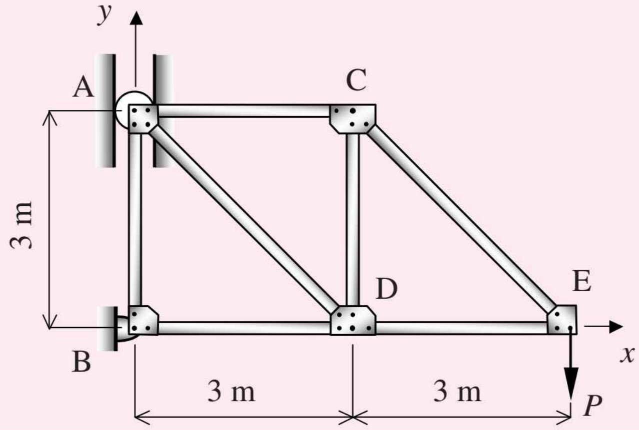
Determine the centroid of the homogeneous composite body.



Exercise 13.45

The frame shown comprises of seven homogeneous members, each having a mass of 100 kg/m.

a. Locate the position of the center of mass.
b. Determine the reaction force at A and B if force $P = 50$ kN is applied at E for the frame to be in equilibrium. Neglect the mass of the gusset plates at the joints.
c. State when analysis of trusses can be applied to solve the problem.



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